A

Historical Introduction

to

Quantum Theory

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Preface

Quantum theory is a branch of physics that has been developed in order to explain new physical phenomena that occurred especially at the end of XIX century. Quantum mechanics is usually felt difficult because of the nature of mathematics used in it. Many sophisticated mathematical tools, some are very abstract, have been forged by brilliant minds during the first quarter of the XX century and thereafter, to understand atoms and also to explain the behavior of elementary particles and their levels of energy inside the atom... the task was not easy.

There is no science without evidence. In the science of physics, experiments are an essential part of evidence. So during the current of those events that led to quantum mechanics, the work done in laboratories played a crucial role in making the new quantum theory. Famous experimenters like **Thompson**, **Rutherford**, **Soddy**, **Mosley**, **Marsden , Geiger, Compton**, **Millikan**,… have opened the door to theorists to emit their theories then they also played a big role to confirm them by another experiments. There have been always good examples in the history of science of complementary effort of theoretical and experimenter physicists.

While quantum theory was forming, Gottingen was the world's center of research in mathematics in the hands of German David **Hilbert** and it played the same role in physics as well. Then another center of physics was founded in Copenhagen (Denmark) mainly because of the work pioneered by Niles **Bohr,** who gathered around him many participants in the new theory like : **Einstein**, **Heisenberg**, **Schrödinger**, **Born**, **Jordan, Pauli**,… so quantum physics is mainly an European theory.

The quantum theory is still continuing its task to find out about particles and interactions. It is also investigating what is going on there out in our big Universe. For example it is present in all the efforts that aim at analyzing the components of galaxies and understand black holes. Quantum theory has also come with the *Theory of Everything* that was proposed to unify all the forces and understand better our Universe and its history. One candidate of this big dream is string theory which has given a lot of mathematics, a lot of hope, but at the same time it has been also a source of much skepticism lately.

This is not a systematic course in physics. The aim of this short essay is to give a concise general view the physics related to quantum theory. To understand the genesis of quantum mechanics, so to give just a glimpse about the ideas before quantum theory, and that chapped the quantum physics since, but to access to that goal one has to give a preparative part to the developments that precedes the invention(or discovery?) led of the new physics.

I think anyone who has finished high school can read the book, I tried to explain the concepts as simply as possible. I tried at first to follow the history but unfortunately this would be not possible, unless the text becomes boring.

There is much information which cannot be included here, so it is not a history of quantum mechanics, I cannot include all the details the creation of quantum theory. Jagdish **Mehta**$(1931-2008)$ has coauthored a book in 6 volumes to the different aspects of the history of quantum theory and perhaps it is not enough. And the most important and decisive reason is the lack of competence.

As usual every chapter is ended with a set of exercises to incite the reader to participate in the course of ideas. These exercises are of great help to the reader to understand the contents. And why not, maybe this could tease his curiosity to go further and maybe try to deepen more into one of the certain part of this beautiful theory.

The text is still being updated,

any mistakes occurring here are mine

Symbols

$\circ$ scalar product.

• usual product between numbers, sometimes omitted.

$\times$ vector product.

$\frac{\partial u}{\partial t}$ derivative of u to the independent variable t, sometimes written simply u_t.

$log(x)$ natural logarithm of x , sometimes noted $ln(x)$.

$\wedge$ exterior product.

$\propto$ proportional to

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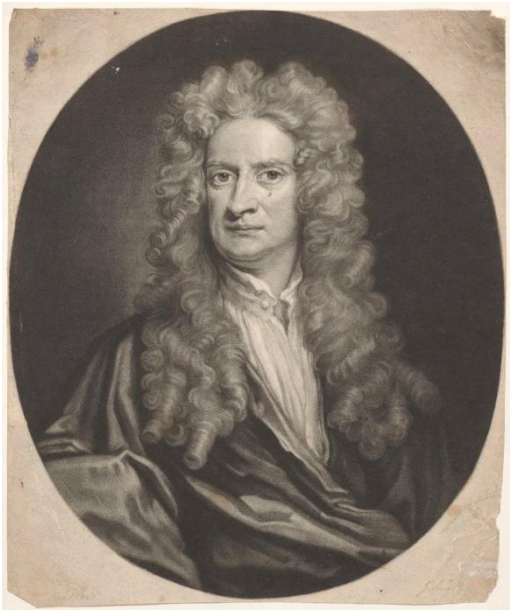
THE NEWTON LEGACY



I do not know what I may seem to the world.
But as to myself I seem to have been only like
a boy playing on the seashore and diverting myself
now and then in finding a smoother pebble or
a prettier shell than the ordinary, whilst the
great ocean of truth lay all undiscovered before me.

Isaac ***Newton***

Mechanics is the part of physical sciences that study the motion of bodies(which we will sometimes call particles for simplicity), it can be applied(practical) or theoretical. Mechanics is classified into three parts: *dynamics*, *kinematic* and *static*. Dynamics has a prominent role in physics, it occupies itself with forces causing motion, to get to kinematics we pass first by dynamics. Dynamics gives us the equation of the motion of the particle. Long before our time, some of mechanics concepts have been mentioned and discussed by Greeks scientists who lived as early as **Aristotle**. **Archimedes** has already discovered the law of the lever

$$F_1 l_1 = F_2 l_2$$

During the Middle Ages and up to the beginning of Renaissance, many physicists belonging to Islamic world or to the European *scholastic* school, continued their work to understand motion and its components, by adding concepts of which we mention:

– Summing forces with the rule of the parallelogram.
– Speed, and the theory of *impetus* (linear momentum)of **Buridan**…
– Static(balanced lever,…).
–Rule of Mertor…

The early results in mechanics before **Galilei** were got mainly by English scientists[1] like William **Heytesbury**$(1313-1372)$, Richard **Swineshead**$(1340-1354)$ and John of **Dumbleeton**$(1310-1349)$, their work remained until the Renaissance, Italian **Di Vinci**$(1452-1519)$ did mechanical constructions with incomparable genius. After all these came also many to continue to free concept from **Aristotle** obsolete dynamics and to find the right theorems.

Galilei

The early one of those people was Italian Galileo **Galilei**$(1564-1642)$ who is often called the father of modern science. With his observations, he tried to formulate laws about motion, learned new things and began to free mechanics from ancient mistakes. Many such wrong assertions must be discarded like of the impossibility of vacuum ,the difference between "weight" and " mass",... He helped in the elucidation of many concepts of the motion of bodies. The motion of *rigid* bodies has an important place in mechanics, a rigid body is a body whose every parts are at a fixed position to each other, it has the tendency to maintain shape. **Galilei** studied falling bodies and the two types of motion : accelerated and uniform. Among his exploits we find:

–In ballistic, he showed that the trajectory of the projectile is a parabola.

–He studied the movement of pendulum and discovered that the period $\frac{1}{\nu}$ is independent of the mass m,

[1] See the books of French Pierre **Duhem**$(1861-1916)$: *l'Evolution de la Mécanique*, *Les Origines de la Statique*, see also *Essay of the history of mechanics* by Clifford **Truesdell**$(1919-2000)$.

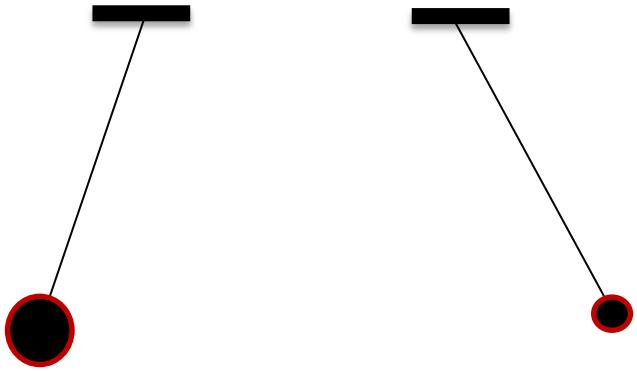


Fig. pendulum with same l and different masses.

—He thus refuted the assertion that since **Aristotle** says "heavy bodies fall before light ones", but all have same acceleration.

—To understand the acceleration of falling objects, he also observed that the motion is similar to the motion of a body on the *inclined plane* . It is obvious in our modern treatment, because when the angle θ increases from 0^o to 90^o and the acceleration from $a = gsin(0)$ to $a = gsin(90)$, the acceleration becomes： $a = g$.

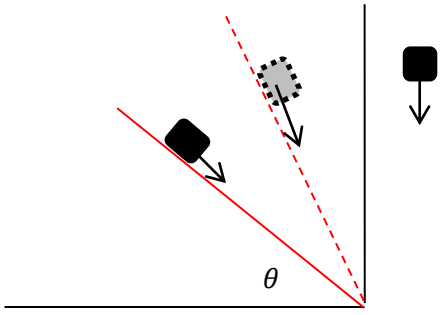


Fig. Changing of the angle

—He helped to introduce time t as a part of the equation of motion, as a physical time.

—He thus inferred that the distance y is in terms of speed v as $y \sim v^2$, It also means that y is for time t as $y \sim t^2$.

—He concluded also that movement is relative and cannot be absolute, a theory which is called **Gallilei** relativity.

Many of his new ideas were expressed in a book published in 1638, four years before his death under the title : *Discourses and Mathematical Demonstration Relating to two New Sciences*[2].

The contemporary of **Gallilei**, German Johannes **Kepler**$(1571-1630)$did important work on the motion of planets. French René **Descartes**$(1596-1650)$ defined what we call nowadays : the *linear momentum*

$$p = mv$$

(physicists of the time did not make a difference between" speed" and "velocity") and he announced that momentum is conserved, one of the early examples of the idea of conservation. **Descartes** too, as did **Gallilei**, announced the inertia law. After that, the known scientist and philosopher German scientist Gottfried **Leibnitz**$(1646-1716)$ introduced the word "*dynamics*". In 1686 , **Leibnitz** rejected **Descartes** claim, and considered instead of quantity of motion $p = mv$ the quantity

$$p = mv^2$$

and called it *force of motion* or *vis viva* and claimed that it is, unlike p , a conserved quantity. So during this time, many concepts began to be studied and understood in modern sense. Like: energy(also called force or live force), momentum, acceleration, …The case where the idea of conservation came to be more known was the collision of bodies, two types of collisions were specified :

—Elastic collision: in the modern language, it implies no loss of total kinetic energy.

—Inelastic collision: it implies a loss of total kinetic energy.

Dutch Christian **Huygens**$(1629-1695)$ was interested in the law of collision between bodies, he showed that the quantity $p = mv$ in collision between two bodies is conserved, namely that

[2] There are translations of this work , as the English translation made by Stillman **Drake** (Wisconsin University Press, London-1974).

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

He also came with his three principles:

- The inertia law, which was not new.
- Conservation of the momentum in elastic collisions.
- relativity of motion(as **Descartes**, **Galilei**).

Although all these efforts by those men have really helped to free physics from old ideas and the road to see the real beginning of classical mechanics, we must wait **Newton** with his laws and big insights.

Newtonian laws

Galilei , **Descartes** and **Huygens**,… with their genius and exploitations, played the role of a transition period from scholastic physics to what is known as "classical mechanics". It was English scientist Isaac **Newton** $(1642 - 174)$ born in the year **Galilie** died, that made possible classical mechanics, also called Newtonian mechanics. **Newton** is known to be the coauthor of calculus with **Leibnitz** , his original ideas were put in his famous book[3] *Philosphiae Naturalis Principia Mathematica.* **Newton** founded the science of dynamics upon three postulated laws :

- *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*
- *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*
- *To every action there is always opposed an equal reaction, or the mutual actions of the two bodies upon each are always equal, and directed to contrary parts.*

The principal parameters in the motion of a particle are : position x and linear momentum p. The first law is not new as we know, it is the *principle of inertia*. The Latin

[3] In English : *Mathematical Principles of Natural Philosophy* published in Latin in the year 1687. A recent English edition was published in 1934 in two volumes, revised by Swiss Florian **Cajori**$(1859 - 1930)$.

word *inertia* means idle, the principle is explained in a dictionary as :

A propriety of matter by which something that is not moving remains still,… it is the propriety of a body by a virtue of which it opposes any agency that attempts to put it in motion or, if it is moving, to change the magnitude or direction of its velocity…

Because of inertia, it is difficult to move a rock from its place by dragging. The third law is formulated mathematically by the equality :

$$F_{12} = -F_{21}$$

where F is force. The unit of force is named newton(N) in **Newton**'s honor. The third law is equivalent to the conservation of momentum. The second law is the *fundamental law of dynamics* :

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = ma$$

it means that the change in velocity of a moving object equals to $\frac{F}{m}$ and this seems logical, for the change in velocity $\frac{dv}{dt}$ of an object can be affected only by the mass of the object and the applied force on it. In current language *weight* is *mass*, but in physics there is a difference between the two; mass is the amount of the matter in an object and weight is the force of attraction (force of gravity) exerted by earth on the object. But they are proportional

$$\frac{F}{m} = const$$

The constant depends on the planet, on earth it is approximately[4] $9.81\ N$. So the weight of an object changes, but its mass never.
The first law is the consequence of the second one. The second law can be written as :

[4] On earth is changes according to. This fact was first noticed by French astronomer Jean **Richer**$(1630 - 1696)$ in 1671. Gravimetry is that interested in the measuring the strength of weight or gravitational field on the surface of earth.

$F = \frac{dp}{dt}$, where $p = mv$ is the linear momentum, a very important factor of bodies' movement.

All the parameters of a movement we will see are measured by three units:

$$\text{meter}(m), \quad \text{Kilogram}(k), \quad \text{second}(s)$$

and every mechanical quantity has a dimension from

$$\text{length}(L), \quad \text{mass}(M), \quad \text{time}(T)$$

so it is in terms of the three above units that we any mechanical quantity. Of course, it would be desirable that all people use the same units to evaluate the physical quantities, but alas this has not been possible for many reasons. The first modern system of measurements was French. In 1791, French scientists were the first to define the modern meter as the unit of length.

The units of the dimensions here are evaluated in the modern metric system called *Système International*(SI) noted also by MKS, where (M) stands is meter, (K) for kilogram and (S) for second[5]. So, the newton, the unit of force is equivalent to : KM/S^2. An ancient unit of length was the cubit which equals 18 inches, it is mentioned for example in the Bible.

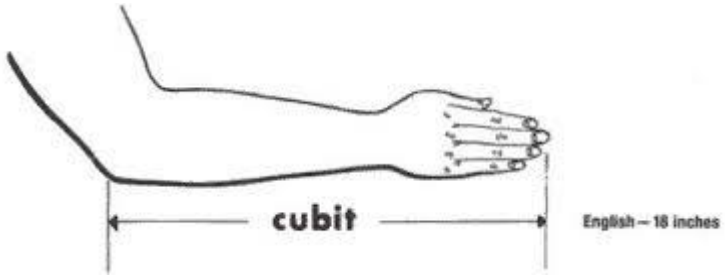


Fig. the cubit unit.

English Henry **Atwood**$(1731 - 1810)$ invented his **Atwood machine** to implement the laws of Newtonian mechanics and evaluate acceleration.

[5] The *Système International* was in 1960. There is another current metric system called the CGS system or the Gauss system. It is also called the centimeter-gram-second system. Here , we use all the time the MKS system.

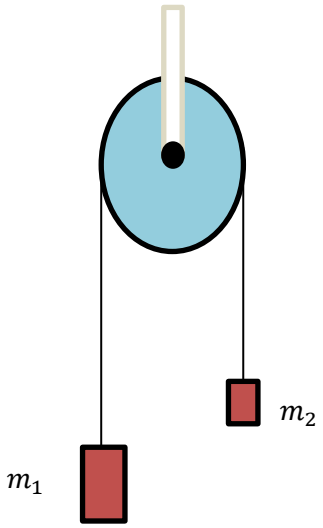


Fig. **Atwood** machine with different masses : $m_1 \neq m_2$

Any force F acting on a body(or *a particle*) is a *vector* that has a magnitude and a *direction*. In a space of n dimensions, a vector has n components $A(a_1, a_2, \dots, a_n)$. In our usual space where we do physics, there are three dimensions, we can write a vector as

$$A = a_1 i + a_2 j + a_3 k$$

or $A = a_1 e_1 + a_1 e_2 + a_3 e_3$. Many physical parameters are vectors like force $F(t)$, position $r(t)$, velocity $\mathrm{v}(t)$ …. We usually endow a vector with an arrow above it : $\vec{A}$, but for simplicity we usually omit it. Other important physical parameters are scalars like: temperature T, mass m,... we will encounter many of these along the text.

Scalar	*Vectors*
Energy, temperature, charge, density, mass, speed, heat,	Forces(weight, …)Displacement, velocity, acceleration, weight, angular momentum, torque, magnetic moment, spin

Fig. Examples of scalars and vectors in physics.

The magnitude of a vector is scalar(in this case a positive number) it is also called :the *norm*, or *module* , it is noted ‖ ‖ so we have:

$$\|F\|, \qquad \|r\|, \qquad \|v\|, \ldots$$

or $\|\vec{F}\|$, $\|\vec{r}\|$, $\|\vec{v}\|$. When a vector function depends of variable $x, y, \ldots$ it is called a *vector field*. Examples of vector field are

$$F(x, y, z) = xi + (y - x)j + y^2 zk$$

and

$$F(t) = ti + (t^2 + t)j - t^2 k$$

A vector field has magnitude ad direction at every point where it acts. An example of vector field is *velocity field* in fluid dynamics, or *electric field* in electromagnetism…
The collection of all formulas of mechanics highly formulated in calculus language as we know them today, is due to the great mathematician and physicist Leonhard **Euler**$(1707 - 1783)$. In fact, and without exaggeration, we could not imagine our time mathematics and physics without the efforts of **Euler**. He modeled all the laws of physics done by physicists before him like **Galilei**, **Huygens**, **Newton**,… in a flexible mathematical language…. In 1736, he published his book *Mechanica,* an important(still not yet translated into any language!!!) modern book in mechanics in two volumes written in Latin language.

Kinematics

Kinematic is defined by physicists as the part of mechanics that studies the *geometry of motion*. In the science of kinematic we establish the relation between many ingredients of a particle motion like: distance(x) , time(t), speed(v), and acceleration(a), through the following equations:

- $x = \frac{1}{2}at^2 + v_0 t + x_0$
- $v - v_0 = at$

– $v^2 - v_0^2 = 2ax$

All these equation stem from the integration in the second law of **Newton**. When $a = 0$ the motion is *uniform* and the above equations become:

– $x = vt + x_0$

– $v = v_0 = constante$

An example in which we can use dynamics laws to find acceleration of a particle is the **Gallilei** example of a body sliding without friction on an inclined plane. So if we take in our example an angle α of 30^o as in the figure :

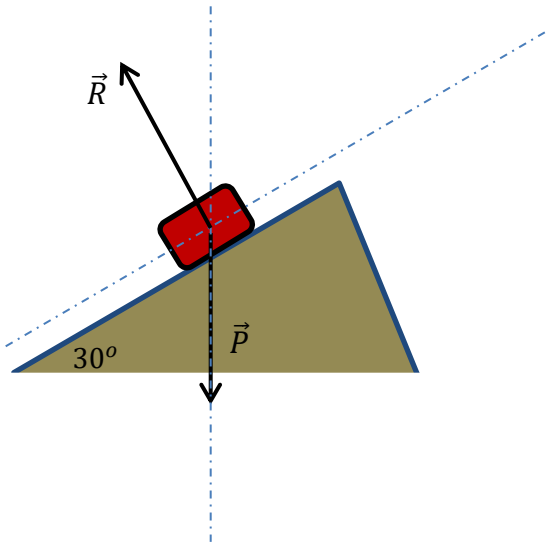


Fig. An inclined plane with $\alpha = 30^o$.

using the second Newton law : $\vec{P} + \vec{R} = m\vec{a}$, and talking a projection on one of the orthogonal axes, we find : $m \bullet g \bullet sin(30) = ma$, so it follows that : $a = \mathrm{g} \bullet \sin(30) = \frac{g}{2}\ m/s^2$. We remark that the acceleration a depends on α and does not depend on m.

Origin of forces

The equation $F = m\frac{dv}{dt}$, and many other equations we will see, is an example of functional equations(where the unknown is a function) called *differential equation*. Differential equations are extremely important in science, they are considered as models to natural phenomena and without them we cannot deal quantitatively with any phenomenon in order to understand it. After a scientist has found the model, he needs to solve it, a task which is not always easy[6]. The above equation is an *ordinary differential equation* (ODE). Any ODE is of the form

$$P(x, y, y', \dots, y^n) = 0$$

and its solution y represents the law we are searching for in the form of an equation $y = f(x)$. A simple ODE is $y' = k$ whose solution is the line $y = kx + b$. One important equation is when $n = 2$ whose solution is the form

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

or

$$y = acosh(c_1 x + c_2)\,, \qquad y = c_1 cos\left(\sqrt{\lambda} x\right) + c_2 sin\left(\sqrt{\lambda} x\right)$$

We will met these solutions often.

We have seen that If $a = 0$ the movement is *uniform*, this means that the speed is constant. It means also that the force $F = 0$. Physicists could not understand what was the origin of the force F, but they believed that it must have a mathematical origin, so they

[6] For example, 250 years after **Euler** had written his original equations modeling fluid dynamics, mathematicians still cannot find the solutions. Another similar equations are the important Navier-Stokes equations.

postulated the existence of a law or a mathematical function V which they called *potential* ("force", "might" in latin). The force is derived from V, thus the **Newton** second law becomes

$$F = -\nabla V(X)$$

where the symbol ∇ is called *nabla* or *del*. So ∇V represents a differential operation (or better an *operator*)called *gradient* (in one dimension, it is simply just the derivative). The gradient transforms a scalar function into a vector. Gradient is a vector field. So the hypothetical V is a scalar function, and in 3 dimension space we have :

$$\nabla V = \frac{\partial V}{\partial x_1} i + \frac{\partial V}{\partial x_2} j + \frac{\partial V}{\partial x_3} k = F_x i + F_y j + F_z k$$

∂ is the partial derivation. The symbol ∇ has other uses that we will see soon. Any force that stems from a potential is called a *conservative*[7] force, and the vector field is *irrotational*.

Falling bodies

The physics of falling bodies was studied by **Galilei**. When a body falls to the ground, it is because of the attraction force of the earth F_g, this force is called *weight* and it is different from the mass m as we have seen. So according to the second law of **Newton** we have $F_g = ma$, we consider $a = const$, as both F_g and m are constant. It is for this reason that a is called the *acceleration of gravity* (or gravitational acceleration) which we note g and we get:

$$F_g = mg$$

In fact, the acceleration g is not exactly a constant and it depends on the Earth force of gravitation, on Earth it changes from place to place, as to

$$9.764 \ m/s^2 \ < \ g \ < \ 9.834 \, m/s^2$$

[7] Like gravitational force as well as electrostatic force. There are some forces in nature that are non-conservative like the force of *friction* in fluids, and the Lorentz force in electrodynamics.

but for convenience, we take its value constant for all exercises and equals to $9.81\ m/s^2$.

When a body falls to the ground, its speed v increases because g is a positive constant. So, the motion of a falling object is an accelerated motion. But the speed v cannot increase without limit, because of the resistance of the air F_{air}(also called *drag*). When F_{air} is negligible, the motion is called *free fall*.

So by solving the differential equation $F_g = mg = m\frac{d^2y}{dt^2}$ we get all kinematic laws as before.

- $y = \frac{1}{2}gt^2 + v_0t + y_0$
- $v - v_0 = gt$
- $v^2 - v_0^2 = 2gy$

So free fall is an accelerating movement with constant acceleration $a = g$.
Since **Galilei** there has been a problematic distinction between two masses : the *gravitational mass* m_g and the *inertial mass* m_i. Inertial mass m_i is the measure of resistance of a body against any change of velocity(the body being accelerated), so

$$m_i = \frac{F}{a}$$

gravitational mass measures the reaction to gravitational field :

$$m_g = \frac{F_g}{g}$$

So weight is not properly mass but depends on mass. The concept of mass[8] is very related to gravity, the attraction force by earth on a body. So the question is :

$$m_i = m_g \quad ?$$

[8] We will see that mass is intimately related to the Higgs boson.

This is the *equivalence principle.* Many experiments (up until now) has been made to measure the difference between m_i and m_g , until Hungarian Loránd **Eötvös**$(1848-1919)$ made his experiment with the *balance pendulum* in 1889 to get a good difference of the order 10^{-9}. The equivalence between the two masses will led to the *equivalence principle* (in special relativity and general relativity) as we will see.

Exercises

1) indicate three forces in nature.
1) show that the second law of Newton is the consequence of the first law.
1) show that $y = e^{-t}(c_1 \sin(t) + c_2 \cos(t))$ is a general solution of $y'' + 2y' + 2y = 0$.
Determine the particular solution such that $y = -2$ and $y' = 5$ at $t = 0$.
1) solve the differential equations:
i) $y' = 2x + 3$
ii) $y'' = 2x^2 + 3x$
iii) $y''' = 2x^4 + 3x^2 - 5$

1) What is force that gives a body of $75\ g$ a speed of $16\ m/s$ in 10 seconds.
1) the position of a particle is given by the equation $x(t) = 1 + t + t^2$. Find the mean speed between time $t = 1$ and $t = 3$.
1) the position of a particle is given by the equation $x(t) = 1i + tj + t^2k$. Find the instantaneous speed at the time $t = 3$.
1) a body moves by the law force $F = ti + t^2j + t^3k$. Find if F is conservative.
1) Find if F is conservative.

i) $F_1 = ti + t^2j + t^3k$
ii) $F_2 = ti + t^2j + t^3k$
iii) $F_3 = ti + t^2j + t^3k$

1) A ball A of $50g$ moving $v = 2cm/s$ hits another ball B of $20g$ at rest. Find the speed of ball B after collision if the speed of A is $v = 0.5cm/s$.
1) find the gradient ∇ of the following scalar functions

i) $x - \sqrt{y} - z$
ii) $x + y - 2z$
iii) $x^3 + y + 2z^2$

1) A ball falling from $y = 5\ m$. Calculate time when it reaches the ground.
1) calculate the value force by which a body of 100 kg hits the ground.
1) calculate the vectors between : $a(1,2,3)$, $b(-3,-2,-1)$ and find its length(norm).

1) calculate the module of the force $F(t) = (t+1)i + (t^2-1)j + t^2k$ at $t = 3$.

1) by integration in the second law of **Newton**, the kinematics laws.

1) a particle is subjected to a potential of the form $V = xyz$. Find F_x, F_y and F_z.

1) a car with $m = 120\ kg$ moving with $10\ N$ on a road without friction. Find the acceleration a.

1) A body with $m = 100g$, the law $r = ti + t^2j + t^3k$. Find the velocity, momentum, and acceleration at the time $t = 7\ s$.

1) A body with $m = 100g$ on the without friction. Find the velocity, after time $t = 7\ s$.

KEPLER LAWS

I have consummated the work to which I pledged myself, using all the abilities that you God gave me, I have shown the glory of your works to men, but if I have pursued my own glory among men while engaged in a work intended for your glory, be merciful, be compassionate, and forgive.

Johannes **Kepler**

Rotational motion is a part of Newtonian mechanics. A classic example of rotational motion is the rotation of Earth around the Sun. However, people, since **Aristotle** (and perhaps before), thought the opposite, that the Sun was turning around Earth, and this was known as the *geocentric theory*. **Ptolemy** on the way of **Aristotle**, devised what is called *epicycles* to deal with the complicated geocentric model, that consisted in explaining the planets paths around the sun. The model remained in action during the middle Ages until Nicolaus **Copernicus**$(1473 - 1543)$ came with his *heliocentric theory* , that is, it is the Sun which is the center, and the Earth turns around it. Although **Copernic** still believed in epicycles as **Ptolemy**, the idea of **Copernic** of heliocentric was taken seriously by **Galilei** and especially, Johannes **Kepler** who, basing on the calculations of his master Danish Tycho **Brahe**$(1546 - 1601)$, got rid of epicycles and announced his three influential fundamental laws that explained the motion of planets around the Sun :

– *The trajectory of a planet around the sun is an ellipse and the sun is at one of the foci of the ellipse.*

– *Every line from the planet to the sun sweeps out the same area in equal periods of time.*

– *The squares of the revolutions period are proportional to the cubes of the axes lengths.*

The three laws can be formulated in mathematical language :

- $r = \frac{a}{1+b\cos(\theta)}$
- $\frac{d\theta}{dt} = \frac{l}{r^2}.$
- $\frac{c^2}{x^3} = const.$

The equation of an ellipse in simple form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it has two foci(plural of focus) and when a tends to b , the two foci unit in one point, the center of a circle and $a = b = r$ becomes its radius,

$$x^2 + y^2 = a^2 \ \ or \ \ b^2$$

so an ellipse is roughly a deformed circle.

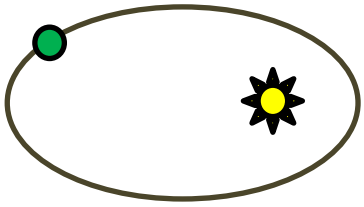


Fig. The elliptic trajectory of Earth around the Sun.
the Sun is situated at one of the foci.

The *Principia* of **Newton** is one of the most influential books in the history of science, it contains his three laws of dynamics we have seen, and also a lot of definitions and theorems of mechanics like dynamics, fluid mechanics,... The results of the book were not written in the proper language of calculus he had invented, but with geometrical language and methods.

The third part of *Principia* is" *On the System of the World*" that contains also the planetary *Law of universal gravity* that consists in the law of the attraction that takes place between

any two bodies of masses m_1, m_2 like two planets. So two bodies attract each other by a force F given by the expression

$$F = G\frac{m_1 m_2}{r^2}$$

the number $G = 6.67 \bullet 10^{-11}\ m/kgs^2$ is the gravitational constant. It follows from this law that the Moon attracts the Earth and the Earth attracts the Moon. Why does not the Moon fall on Earth? Because the attraction is weak, and there is another explication linked to its speed and the centrifuge force.

The **Kepler** laws helped **Newton** to formulate his ideas on mechanics, so they were the basis by which he could arrive to his marvelous system. It follows from **Newton** law of attraction that acceleration is the same for all falling bodies, a result already known to **Galilei**. **Newton** first viewed the idea that forces between bodies can act by action at distance unacceptable, then he changed his mind.

The Newton attraction law was established afterwards by English physicist George **Cavendish**$(1745 - 1807)$ in 1797 by measuring the constant G. The gravity force seems to have been known to some scientists before **Newton**, like Seljuki scientist Abdurrahman[9] **Al-Khazini**$(1115 - 1155)$.

Proprieties of rotational motion

The speed of rotating object around a center is *angular* and usually noted ω, it has a relation with v through the relation $\omega = \frac{v}{R}$ taken from the fact that $R\alpha = l$ (we know from geometry that $\frac{l}{R} = 2\pi$). The *angular acceleration* is given by the relation $\omega' = \frac{d\omega}{dt}$. When the movement is rotational like the movement of the Moon around the Earth, there are two accelerations : *tangential* a_T and *normal* a_N :

[9] He wrote many books in physics and mathematics, one of them was published in the *Journal of the American Oriental Society*(1860).

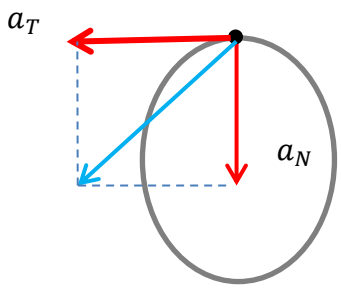


Fig. a_T and a_N .

–The tangential acceleration a_T represents the rate of change of the speed, it has the expression : $a_T = \frac{dv}{dt}$.

–The normal acceleration a_N , which presents the change in the direction of the velocity. The normal acceleration hints towards the center, it has the magnitude expression $a_N = \frac{v^2}{R}$.

And we have the total acceleration vector of an object turning around a center O given by the formula :

$$a = a_T + a_N$$

To make a body rotate, we apply on it a suitable force F and we define the *torque* τ (from Latin *torquere* : to twist) which is defined as

$$\tau = F \times l$$

l is the length of the arm. Particularly the bigger is l , the easier it will be to rotate an object, and this is seen through the relation

$$|\tau| = |F||l|sin(\theta)$$

The torque is “the measure of the turning effect produced on the body by the force” .

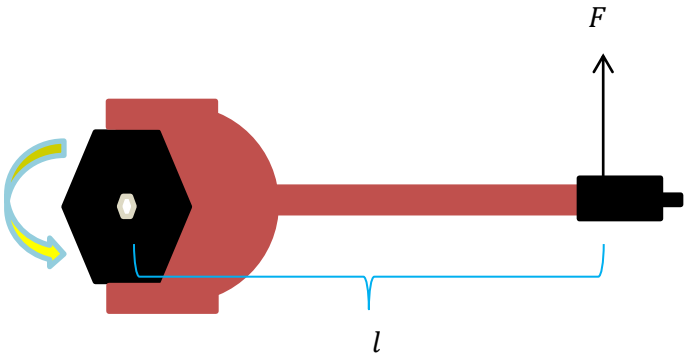


Fig. τ is perpendicular to the plane of F and l

the notation $\times$ is the *vector product*. Given two vectors $A(a_1, a_2, a_3), B(b_1, b_2, b_3)$, the vector product is defined by

$$\mathrm{A} \times \mathrm{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3 - a_3b_2) - j(a_1b_3 - b_1a_3) + k(a_1b_2 - b_1a_2)$$

$\mathrm{A} \times \mathrm{B}$ is itself a vector, it is perpendicular to the plan formed by the vectors A and B. We have two useful relations

$$\nabla \times (\nabla \mathrm{u}) = 0$$

$$\nabla \circ (\nabla \times \mathrm{u}) = 0$$

Torque is thus a vector too, called also *moment of force*. In case of rotational motion the **Newton** second law is

$$\frac{d\mathcal{L}}{dt} = I\omega'$$

where $\mathcal{L}$ is the *angular momentum* defined by

$$\mathcal{L} = r \times p$$

The vector p is the linear momentum $p = mv$. I is the *moment of inertia*, it is generally a tensor(we will see tensors later) and it takes the place of m in rotational motion. We can read in a dictionary

There are two numerical measures of the inertia of a body: its mass, which governs its resistance to the action of a force, and its moment of inertia about a specified axis, which measures its resistance to the action of a torque about the same axis.

Torque measures also the rate of change of the angular momentum

$$\tau = \frac{d\mathcal{L}}{dt}$$

so the similar law to **Newton** second law in the case of rotational motion is

$$\tau = I\omega'$$

the where torque angular momentum and torque appears is the gyroscope

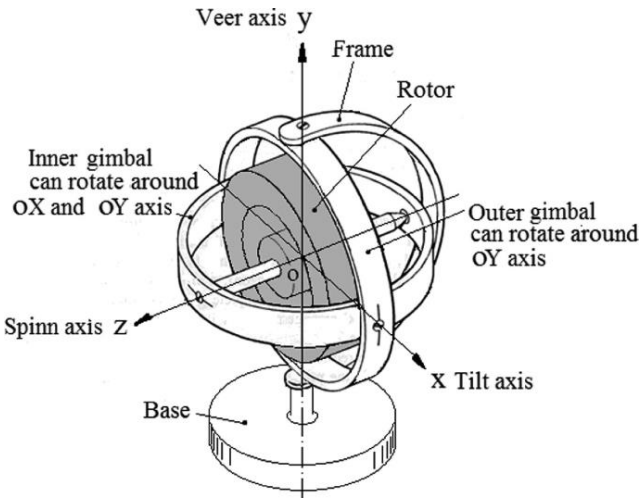


Fig. gyroscope.

In 1852 , French Léon **Foucault**$(1819-1868)$ gave another proof of the rotation of earth(after his famous pendulum) using a gyroscope.

It would be very convenient if we make a comparison between linear and rotational movements by giving the factors playing the same role:

Linear movement	*Rotating movement*
m	I
v	ω
$F=-\nabla V$	$\tau=\frac{d\mathcal{L}}{dt}$
$F=ma$	$\tau=I\omega'$ $F=mR\omega^2$

Fig. Comparison between the parameters of both linear and rotational motions.

Classical features

When one has studied the development of quantum theory, and want to make a comparison with classical mechanics he can see some characteristics special to each one of the two mechanics, so some remarks are in order here. For instance, classical mechanics is *deterministic*,…

Newton mechanics has been not saved from criticism before the advent of quantum mechanics. Austrian Ernest **Mach**$(1838-1916)$ in his book *The Science of Mechanics*(1883) criticized **Newton** for many things, for example for defining mass as: $m=density\bullet volume$, which he considered as a vicious circle, also for his absolute time and space…. . but despite all this, **Galilei** and subsequent physicists have really accomplished a big work founding classical mechanics. The mechanics of **Newton** is based on : force, mass, space and time. I want here to analyze some of the proprieties of this mechanics in such a way that a comparison can be made in the future between it and quantum mechanics.

–*Classical principle of relativity:* every motion is relative to another motion. There is no absolute motion. And all the laws are the same in inertial frames where the motion is uniform. So a speed of a body in accelerated motion is relative and not independent.

–*Causality* : roughly it is the statement that says : everything has a cause that produces it. The cause of motion is force. This has deeper philosophical meaning in some aspects of modern physics. For example we say that a particle A sends a message in the form of a signal to B then B responds by sending a message in the form of another signal. Now can one imagine that B sends his respond to a message sent by A not yet received ? or never send by A?

It is said that physicists(atheists or not) should prove physics theorems by scientific methods only[10]. They believe that this is the essence of doing science, so that every event has a cause and the law governing it can be formulated by the language of mathematics is the rule. They believe that the method of science can explain, they do not consider the intervention of God, or whether he has created these laws. This is not new, traces of this sort of thought exits in the $XVII$ century. French Jean **D'Alembert**$(1717-1783)$ says on *the principle of final causes*:

The laws of equilibrium and motion are necessary truths. A metaphysician would perhaps be satisfied to prove this by saying that it was the wisdom of the Creator and the simplicity of his intentions never to establish other was of equilibrium and of motion than those which follow from the very existence of bodies and their mutual impenetrability. But we have considered it our duty to abstain from this kind of argument, because it has seemed to us that it is based on too vague a principle.... we can only discover the effect of his wisdom by the observation of the law of nature, since mathematical reasoning has made the simplicity of these laws evident to us, and experiment has shown us their application and their scope. it seems to me that this consideration can be used to judge the value of the demonstration of the laws of motion which have been given by several philosophers, in accordance with the principle of final causes; that is, according to the intentions that the Author of nature might have formulated in establishing these laws. Such demonstrations cannot have as much force as those which are preceded and supported by direct demonstrations, and which are deduced from principles that are more within grasp.

But this will not solve the problem, for this debate can raise two questions: what is science itself? and can science prove everything ? we will encounter other questions of the same nature.

–*Locality* : The word locality, in this context means nearness. It means that there is no instantaneous effect of a body A on another body B. A body can be affected by another body only on its immediate surroundings. **Newton** thought that gravity cannot work without

[10] Even proofs can be subjectively different. The great German mathematician David **Hilbert** proved his basis theorem in invariant theory by a non-constructive method, his fellow German Paul **Gordan**$(1837-1912)$ remarked :" *This is not mathematics , this is theology*".

a medium so the he postulate the existence of ether. He says

It is inconceivable that inanimate brute matter should, without the mediation of something else which is not material, operate upon and affect other matter without mutual contact

locality will see another turn with the interpretation of quantum mechanics.

–*Determinism:* in the mechanics of **Galilei** and his followers, we can totally predict the development of the motion of a body. So if a bullet is shot and we know the rule of its ballistic motion, we can tell exactly when and where the bullet drops. If we know all the information (speed, air resistance,…) of motion we can *determine* its parameters at any time. The equation of the motion determines the motion of the particle. Determinism can be also called *realism* in the quantum world. It is strict causality. We notice that there is an anachronism, so for example the word *determinism* was coined only after quantum mechanics had been finished as a theory and the problems around its interpretation were discovered.

Exercises

1) show that the acceleration is the same for all bodies on earth(use the second law with the law of attraction).

1) calculate the attraction force of the Earth on the Moon.

1) Show that $a_N = 0$ when the trajectory is a line.

1) Show that $a_N = 2$ and $a_N = 3$. Find the acceleration a.

1) Show that $a_T = \omega' R$

1) Show that $\mathcal{L} = I\omega$.

1) Show that $\int_{t_1}^{t_2} F dt = m(v_2 - v_1)$. The action.

1) if $\mathcal{L}(t) = 1 + t$ find the torque τ at $t = 2$.

1) a body moves by the law $r = xi + y^2 j + z^3 k$, and the force $F = (x + y)i + t^2 j + t^3 k$ Find the torque τ .

1) Calculate the impulse of the force given by : $F = (x-1)i + y^2 j + z^3 k$ between $t = 1$ and $t = 2$.

1) Show that $\tau = m(r \times \mathrm{v})$

1) a car moving in a circle with $R = 200m$. If $v = 50m/s$ find :

−angular speed.

−angular acceleration.

−normal acceleration.

1) Show that if the force F is conservative then we have $\nabla \times \mathrm{F} = 0$.

1) show that $\mathrm{A} \times \mathrm{A} = 0$.

1) calculate the vector product $\times$ of the following vectors in each case:

i) $i - 2j - 3k$ and $1i - 2j + k$

ii) $2i - 2j + k$ and $1i - 2j - 7k$

iii) $i + j + 3k$ and $1i - 2j - zk$

1) Let the vector : $ti - 2^t j - t^3 k$. Calculate $\nabla \circ, \nabla \times$.

1) show that $|A \times B| = |A||B|sin(\theta)$.

1) Calculate ∇ for scalar : $u(x, y) = xy - y^2$.

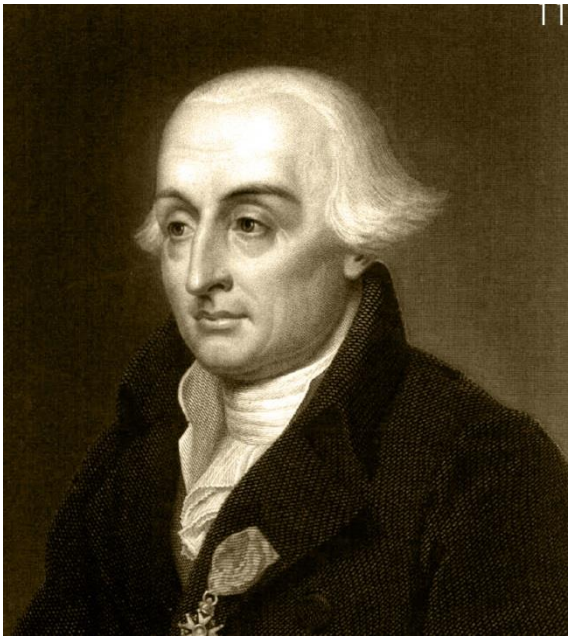
1) Then find $\nabla \circ (\nabla \times \mathrm{u}), \nabla \times (\nabla \mathrm{u})$.

1)when is the torque maximal? minimal?

1)calculte density(specific mass) of water the if of $1L$ weights $1kg$.

1) we have seen that I is the moment of inertia, and like m , it measures the resistance of the body to rotational moment. What represents the distance x in the case of rotational motion ? Define ω and ω' in term of it.

ENERGY



In general, I believe that I may assert that all the general principles which may still be discovered in the science of equilibrium, will be only this same principle of virtual velocities, looked at from different points of view, and differing from it only in the form of statement.

Joseph **Lagrange**

Analytic mechanics is that part of classical mechanics that uses analysis i.e. the method of calculus to study the motion of objects. After the work **Newton** and **Euler** Other reformulations of the science of mechanics took place. One was due to Joseph-Louis **Lagrange**$(1736-1813)$ and the other to Irish William **Hamilton** $(1805-1865)$. Whereas **Newton** founded his mechanics on forces F(vectors) the above successors founded their mechanics on *energy* (scalars). Energy contains two important parameter : distance (x) and speed (v). Still, the two new laws have been shown to be equivalent to Newtonian mechanics.

The idea of both scientists is to define an *operator* in terms of kinetic energy K and potential energy V. This is not very strange because we can, from Newton second law, find a relation between K and V roughly as follows

$$F=\frac{dp}{dt}=-\nabla V$$

but[11] $p=\frac{dK}{dv}$, which gives

$$\frac{d}{dt}\left(\frac{dK}{dv}\right)+\nabla V=0$$

[11] The fact that momentum p is the derivative of kinetic energy with respect to v is like the fact that the area of a sphere is the derivative of its volume with respect to radius.

hence Newton law implies a relation between K and V.

Classical mechanics in this new context is then the mechanics of **Newton**, **Lagrange** and **Hamilton**, the aim of all of it is to find the equation of motion for a moving body. Modern physics includes two other parts: quantum mechanics and relativity. Although the two mechanics of **Lagrange** and **Hamilton** are both equivalent to **Newton** mechanics, they are simpler and more suitable to generalize to other situations as we will see in the case of quantum mechanics(field theory). We will in many occasions face the two mechanics used by theorists in the XX century to create their theories on the nature of subatomic particles like quantum field theory, quantum electrodynamics, statistical and celestial mechanics.

There is a relation between force, momentum and energy. Physicists also defined the *impulse* j between two points as

$$j = p_2 - p_1 = \int_{t_1}^{t_2} F dt$$

this comes easily from second law $F = m\frac{dv}{dt}$, so by integrating

$$\int_{t_1}^{t_2} F dt = m(v_2 - v_1) = \int_{t_1}^{t_2} \frac{dK}{dv} dt$$

The impulse is related to momentum. So the impulse of forces between two points equals to the change in linear momentum.

Energy

The word energy was introduced by English Thomas **Young**$(1773 - 1829)$ in 1807 to mean "quantity of motion" or mv^2. This means in our modern kinetic energy[12]. Energy is a concept to which it is difficult to find a definition, because it has many facets. It took for the concept of energy a lot of time to be understood. We will first define the concept of *work*.

[12] **Leibnitz** knew kinetic energy under the name of *vis viva* and he thought it equaled to mv^2.

Although physicists had speculations for long on energy and work, both the words : work and energy were introduced only in the XIX century. Also one can notice that they are related as they are measured by the same unit.

through its movement, a body can accomplish a ***work*** , which is defined as the force multiplied by distance

$$W = Fr$$

When the motion is in the plane ($\mathbb{R}^2$) or in the space ($\mathbb{R}^3$), we will need the dot product and the equation becomes $W = F \circ r$. So usually, we look for the work accomplished between two points x_1 and x_2 .

Energy is a word derived from Greek word $\varepsilon\nu\varepsilon\rho\gamma\varepsilon\iota\alpha$ which means "activity". Energy is linked to work, it is "the ability to accomplish work". Energy trough the present text can be presented by many letters:

$$E, \quad K, \quad W, \quad P, \quad V, \ldots$$

this because energy has many forms: mechanical, magnetic, chemical, electric, thermal(heat),nuclear, …but it can be[13] either ***kinetic*** or ***potential***.

– The kinetic energy K is the energy of motion, it depends on speed. Mathematically, kinetic energy can be calculated roughly as

$$\int dw = \int F \circ dr = \frac{1}{2}mv^2$$

The work accomplished between two points x_1 and x_2 is

$$K_2 - K_1 = \frac{1}{2}m({v_2}^2 - {v_1}^2)$$

regarding its definition "the ability to accomplish work", it is intuitive to understand that the

[13] We will see that the terms potential and kinetic have been introduced in the theory of heat by **Rankine** and **Kelvin** respectively.

ability of a body to affect (destruct, or be destructed,…) another body depends on its *mass* m and *speed* v. In the case of a rotational motion it is $K = \frac{1}{2} I\omega^2$. So kinetic energy is always a function of *speed*.

– Potential energy is the form of energy stored inside matter and as an "energy" it is ready to burst out and accomplish work. A body falling from the height h has a potential energy, a *spring* can have a potential energy,… so potential energy may depend on : height x, mass m, charge q, …. The following figure contains different cases :

Case	*Potential energy*
Linear spring	$\frac{1}{2}kx^2$
charge	$\frac{q}{4\pi\varepsilon_0 r}$
gravitational	$-G\frac{m}{r}$
from earth	ghm
nuclear	mc^2, …

Fig. Potential energy in many cases.

Conservation of energy

Many physicists supposed that many parameters of motion were conserved, we have seen the conservation of linear momentum and vis viva. From kinetic and potential energy we can define *mechanical energy*

$$E_{mec} = K + V$$

mechanical from "mechanics", related to the movement of bodies, it also called *total* energy. Kinetic energy can transform to potential and vice versa, potential energy can transform to kinetic energy. A body situated at the height h has a potential energy and able to do work, when it is falling to the ground it acquires a kinetic energy, so the total energy is

a mechanical energy

$$E_{mec} = \frac{1}{2}mv^2 + ghm$$

what is observed is that total energy is conserved[14]. Or , during the fall process the total energy is constant : $K + V = constant$. The pair (E, p) is important in the understanding of the motion of a body, because they are *conserved*. In fact physicists concluded the conservation of three parameters:

- Linear momentum p.
- Angular momentum $\mathcal{L}$.
- Energy E.

An example of the conversation of momentum is the *Newton cradle*.

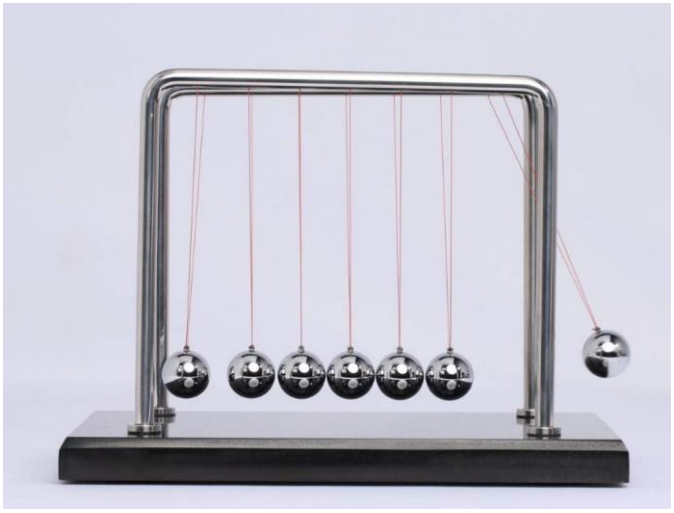


Fig. Newton cradle

From the equation $E_{mec} = \frac{1}{2}mv^2 + ghm$ we can calculate some useful parameters of the motion like v. We take an example: A body from the height $h = 3m$, find its speed when it

[14] In some situations, mechanical energy is not conserved as in the case of *inelastic collision*.

reaches the ground. From the conservation of the total energy, we have :

$$\frac{1}{2}m{v_1}^2 + gh_1m = \frac{1}{2}m{v_2}^2 + gh_2m$$

$$0 + ghm = \frac{1}{2}mv^2 + 0$$

$$v = \sqrt{2gh} = 7.67\ m/s.$$

Notice that we did not need mass. Energy, having many forms, we will see, is conserved and one form of energy can change into another form. We remark that this transformation is done in a continuous manner, unlike in the quantum case.

We will see that energy expression can have a curious form

$$E = \sqrt{p^2m^2 + 4K^2}$$

we will see that the deep cause of conservation is symmetry.

Extremma

A function f may have points where it has a maximum(max) or a minimum(min), each of them is an example of an ***extremeum*** (extrema is the plural). A necessary condition for derivable function f to have a min(or max) x_0 is : $f'(x_0) = 0$, the function $f(x) = x^2$ has a min at $x = 0$ as $f'(0) = 0$.

But the inverse is not true : the fact that $f'(x_0) = 0$ does not infer that x_0 is an extremum for the function f. The function $f(x) = x^{3/2}$ has no extremum even when $f'(0) = 0$. When the function f has more than one variable, the situation is more complicated.

Physicists study also the equilibrium of bodies. An important situation in the science of mechanics where an extremum (a minimum) is implied is in the case of the ***equilibrium*** of a body. A condition for a body to be in to equilibrium is when $F = 0$, so the potential $\nabla V = 0$ and this means that V may have a minimum or a maximum , which is equivalent to say that

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = \frac{\partial V}{\partial x_3} = 0$$

so find the *equilibrium*, and this is verified when

$$\overrightarrow{F_x} + \overrightarrow{F_y} + \overrightarrow{F_z} = 0$$

The case $\overrightarrow{F_x} = \overrightarrow{F_y} = \overrightarrow{F_z} = 0$ is a special case. Physicists can be also interested in the *stability* of a point of equilibrium of a body. One says that the point of equilibrium is stable if

$$\textit{A point } x \textit{ is stable} \iff \textit{The potential } V \textit{ is minimum at } x.$$

one example is the pendulum, when a pendulum is swinging it continuously transforms potential energy to kinetic energy and vice versa, until it rests to equilibrium. So the potential V reaches its minimum. So given a potential $V(x)$, one find first the points of equilibrium, then to find out whether this equilibrium is stable or not.

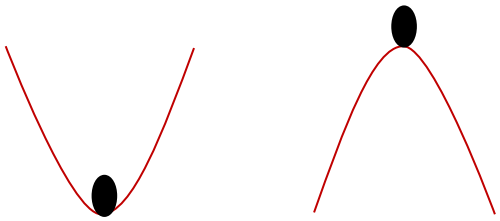


Fig. The equilibrium point of the particle is stable at A and unstable at B.

Lagrange

Around the year 1740, **Euler** posed the question: when a function $f(x, y)$ takes its minimum[15] between two points a and b? he found out that it is necessary that the integral

[15] In 1696, Johan **Bernoulli**$(1676 - 1748)$ posed the following :what is the trajectory curve of a particle going from A to B subjected to gravity only in the least time. It is called *the brachistochrone problem*.

$$\int_a^b f(x, y, y')dx$$

has an extremum. The **Euler** has that the function f to verify his known equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

published in 1744. The equation is an example of a *partial differential equation* (PDE).

At the same year i.e. In 1744 when **Euler** found his equation, an idea came from French Pierre **Maupertuis**$(1698 - 1759)$ who discovered the principle of least action. It consists simply in that fact that a moving body takes always the path with least effort. He said :

L'Action est proportionnelle au produit de la masse par la vitesse et par l'espace. Maintenant, voici ce principe, si sage, si digne de l'Etre Suprême : lorsqu'il arrive quelque changement dans la Nature, la quantité d'Action employée pour ce changement est toujours la plus petit qu'il soit possible.

He means by the expression mvx the action i.e. : energy•time which is in fact mv^2t .

By these considerations **Euler** founded a new part of research called the *calculus of variations*. The calculus of variations is a subject in mathematics that looks for methods that a minimize or a maximize of a function(called functional) like an integral by another function. It is big area of research in mathematics and it use modern tools of functional analysis.

While at the Berlin Academy, **Lagrange** was introducing his equations around 1760 , then he put all his new ideas in his book *Analytic Mechanics*(1788). The chosen method can facilitate the solution of the mechanics problems, he privileged mechanics without graphs and drawings and put all the system on the principle of *virtual velocities* or "virtual work". This idea stems from the law of the balanced lever found by **Archimedes** and existed also in the **Bernoulli** work. The idea of **Lagrange** is to *minimize* a function. A comparable idea is the **Fermat's principle** due to French Pierre **Fermat**$(1600 - 1665)$ to the question:

What is the shortest path taken by light between two points ?

this of course depends on the *medium* between the two points, and the path taken by light is the one in which the *least time* is taken.

Lagrange introduced new concepts:

i) Generalized coordinates: the reader knows what we mean by coordinates of a moving body like : cartesian, spherical, cylindrical (polar), and the parameters are

$$\text{distance } (x) \quad , \quad \text{angle } (\alpha)$$

but the Lagrange generalized coordinates $(q_1, q_2, \dots, q_n)$, of course are not only the two principal(x , α) but can be anything linked to them like : velocities, momentums,… there is a condition : they must be independent. So they can formulate the system completely, and change according to the physicist choice. So, if we have N particles and n coordinates $(q_1, q_2, \dots, q_n)$ without forgetting time t the motion of a particle is represented by

$$(q_1, q_2, \dots, q_n, t)$$

With generalized coordinates we have the *degree of freedom*:

ii) Degree of freedom : A system of N free particles has $3N$ degree of freedom. So

$$(q_1, q_2, \dots, q_n, t)$$

degree of freedom is the number of parameters that completely specify the motion of the system.

ii) velocities and their derivatives $\dot{q}$, $\dot{q}$ where can generalized *velocities* and t is time.

$$(\dot{q_1}, \dot{q_2}, \dots, \dot{q_n}, t)$$

Then Lagrange defined an operator known later as the *Lagrangian* of the system of

$$L = K - V$$

where K is the kinetic energy of the body and V is potential energy, so it is not the total energy.

So to the motion the Lagrangian L must be the solution of the same Euler equation mentioned above, and it is known now as the **Euler-Lagrange** equation as it has also discovered by **Lagrange**. The chosen general coordinate q

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

in place of $f(x, y, y')$ we put L we find the equation of the motion. In order to discover his equation, **Lagrange** defined the *action* S such that from time t_1 to t_2

$$S = \int_{t_1}^{t_2} L dt$$

So[16]

$$action \ = \ energy \ \bullet time.$$

Let's now give an example of to solve a problem in mechanics with the lagrangian: a body falling from a height h , so to find of the equation of the movement we :

- write first the expression of the lagrangian L by finding $K - V$.
- then to substitute L in the **Euler**−**Lagrange** equation to find the equation of the motion.

So we have $K = \frac{1}{2}mx''^2$, whereas $V = mgx$ so

[16] We can also write : action = momentum•distance, so : $S = \int_{x_1}^{x_2} p dx$.

$$L = \frac{1}{2}mx''^2 - mgx$$

Using the Euler-Lagrange equation we can find the required equation

$$x'' = g$$

and the solution is the equation

$$x = \frac{1}{2}gt^2 + v_0 t + x_0$$

Let consider the simple pendulum, our task is to find the motion equation.

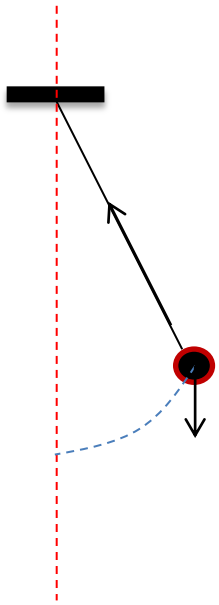


Fig. A pendulum (harmonic oscillator).

We chose as the generalized coordinate, the angle α and the length l of the pendulum. So we have $K = \frac{1}{2}mv^2 = \frac{1}{2}m(l\alpha')^2$, whereas $V = -mgl\cos(\alpha)$ so

$$L = \frac{1}{2}m(x\alpha')^2 - mgl\cos(\alpha)$$

Using the Euler-Lagrange equation we can find the required equation

$$\alpha'' + \omega^2 sin(\alpha) = 0$$

It is a linear ODE. When the oscillations are small, the equation becomes $\alpha'' + \omega^2\alpha = 0$ and the solution is

$$\alpha = Acos(\omega t + \varphi)$$

it is the equation of a simple harmonic motion. An non linear ODE for an *anharmonic* oscillation is

$$\alpha'' + \omega^2\alpha + \lambda\alpha^3 = 0$$

Hamilton

Another theory that it is simpler in dealing with complicated problems and at the same time equivalent to **Newton** method, was found by many physicists at the beginning of the XIX century. It was principally Irish William **Hamilton,** who in 1827 in his paper" *A Theory of Systems of Rays*" gave his new insights to fund a new reformulation of Newton mechanics. In 1834, he said on Lagrange mechanics

…among the successors of those illustrious me, Lagrange has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequence respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem.

in 1833 he introduced the *varying action*, replaced velocity by *momentum.* The idea from which he was inspired came also from French Pierre **Maupertuis** as seen. The principle is that a body in its movement takes the path of *least* action or :

When any change takes place in nature, the quantity of action necessary for this change is the smallest possible.

and this can be by the formula

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0$$

When $\delta S = 0$, we get the **Euler-Lagrange** equation. So as in Lagrange mechanics Hamilton introduced :

i)generalized coordinates:

$$(q_1, q_2, \dots, q_n, t)$$

ii)generalized momentum and their derivatives

$$(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$$

where $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$can generalized ***velocities*** and t is time.

He this time introduced the new operator

$$\mathcal{H} = p_i \dot{q}_i - L = 2K - L = K + V$$

so gave us the ***Hamiltonian*** : $\mathcal{H} = K + V$ which is in this case the total energy. This led us the **Hamilton** equations :

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}$$

In 1837 German **Jacobi** $(1804-1851)$, and the equation is known under the name **Hamilton-Jacobi**[17] equation

$$-\frac{\partial S}{\partial t}=H\left(q,\frac{\partial S}{\partial q},t\right)$$

We remark that the equation is second order, whereas Hamilton equation is of first order. in 1809 before Hamilton mechanics French **Poisson**$(1781-1840)$, a student of **Lagrange** , introduced the *Poisson bracket*[18] . The Poisson bracket of two function f,g is defined as

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Poisson consider the dynamics of planetary attraction after the work of **Laplace** and **Lagrange** on analytic mechanics, the idea is all around the integration of differential equation and the determination of arbitrary constants. We can write Hamilton equation in terms of Poisson bracket

$$\frac{\partial G}{\partial t}+\{G,H\}=0$$

In 1837 German Carl **Jacobi** $(1804-1851)$ gave another equation and he on **Poisson** brackets. **Hamilton** used the principle of least action to solve the **Hamilton-Jacobi** equation

$$-\frac{\partial S}{\partial t}=H\left(q,\frac{\partial S}{\partial q},t\right)$$

which is a nonlinear PDE. We will see that in quantum mechanics, Poisson bracket $\{x,y\}$ will be used too and it will become : $\frac{1}{i}[x,y]$.

[17] See : *The Early History of Hamilton-Jacobi Dynamics 1834-1837* by M. Nakane and C. Fraser.

[18] In his paper :"*Sur la variation des constantes arbitraires dans les questions de mécanique*". We will see that **Dirac** introduces the Poisson bracket in his work.

Exercises

1) give an essential difference between energy and momentum.
1) calculate the potential energy of a body falling at the height $300m$.
1) calculate the kinetic energy of the same body .
1) calculate the potential energy of a spring with $50cm$ and $k =$.
1)Show that

$$\int_{v_1}^{v_2} F \circ dr = \frac{m}{2}({v_2}^2 - {v_1}^2)$$

1) Show that $\|r\| = \sqrt{r \circ r}$.
1) calculate the impulse between two points where that $v_1 =$
1) calculate the energy of a body going in $R = 20cm$ and at the speed of 300 by hour.
1) find the kinetic energy of a particle of mass $20g$ moving with velocity $i + 10j - 20k$.
1)find the all partial derivative for :
i) $x^2 + xy + z^4$
ii) $2\,x^2 + xy - 2$
iii) $zyx^2 + 3x - 1$
iv) $x^2 + \log(xy) + z^4$

1) when a work is min?
1) when a work max?
1) A particle **P** is moved by a force $F = 10i - 20j - 15k$ along a straight line from A to B whose position vectors are $2i + 7j - 3k$ and $4i - 2j - 7k$. Find the work done by the particle.
1) have the following parabolas minimum or maximum?

i) $x^2 + x + 3$
ii) $-\,x^2 + x - 2$
iii) $x^2 - x + 1$

1) a necessary condition for to be min is $y' = 0$. Is $x = 0$ a min for $y = x^3$?

1)find the extrema if any of the following:
i) $x^2 + 2x + x^4$
ii) $2\,x^3 + 6x - 2$
iii) $-x^5 + 3x - 1$

1)find the when a torque a max min
1) let the following potential $V(x) = x^2 + 3x - 1$. Find the points of equilibrium then test if the equilibrium is stable.
1) find the Lagrangian of a body falling freely to the ground.

1)A body of m=10kg moving in path from x_1 to x_2. The speed in x_1 is $10km/h$ an in x_2 is $45km/h$. Calculate the work W done in his motion.

1)A body moving in a linear path from x_1 to x_2. Using the link to linear momentum show that $K = \int_{x_1}^{x_2} F dx$.

1) find the Hamiltonian of system body sliding on an inclined .

1) find the solution $u(x, y)$ for the following PDE

i) $u_x = 2$.
ii) $u_y = 2$
iii) $u_x = x$
iv) $u_y = y$
v) $u_x = y$
vi) $u_y = x$

1) find the **Poisson** bracket for every couple $\{g, f\}$

i) $\{x + y, x - y\}$.
ii) $\{xy, x - y\}$
iii) $\{x + y, y\}$

1) find show that $\{f, g\} = -\{g, f\}$

1) find show that $\{f, f\} = 0$

1) find show that $\{f + g, h\} = \{f, g\} + \{f, h\}$

ELECTROMAGNETISM

Almighty God, who has created man in Thine own image, and made him a living soul that he might seek after Thee, and have dominion over Thy creatures, teach us to study the works of Thy hands, that we may subdue the earth to our use, and strengthen the reason for Thy service; so to receive Thy blessed Word, that we may believe on Him whom Thou has sent, to give us the knowledge of salvation and the remission of sins...

Clark ***Maxwell***

The word electricity comes from the Greek $\eta\lambda\varepsilon\kappa\tau\rho\sigma\nu$ that means "amber", the Ancients knew that amber substance when fractioned with fur can attract light objects. Electricity began as an independent subject when Italian Alessandro **Volta**$(1745-1827)$ constructed the ***Volta pile*** considered as the first battery in history. Magnetism is also known to the Ancients, the word magnet originated from the Greek $\mu\alpha\gamma\nu\eta\tau\iota\varsigma\ \lambda\iota\theta\sigma\varsigma$ that means *lodestones* , a material that has the propriety of attracting iron. Before the XIX century physicists were ignorant of the relation between the apparently unrelated two phenomena of electricity and magnetism.

The nature of light

Waves as material perturbations, are not strange to people. A rope can have a form of a moving wave when shacked by hand,

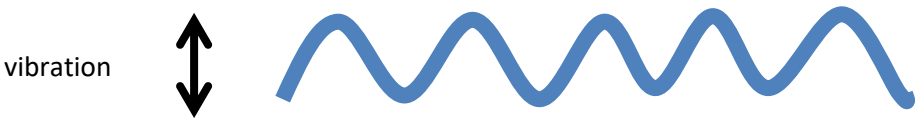


Fig. waves in a rope

waves like those formed in the waters of a lac water when perturbed by a pebble are daily practices.

Fig. water waves

Material waves can be :

- The transverse waves: that oscillate perpendicularly to the direction of motion like : light waves, electromagnetic waves in general, water waves, string waves.
- The longitudinal waves : that oscillate in the same direction of the motion like : sound waves.

During the $XVIII$ century physicists did not agree on the nature of light. **Newton** thought that light was composed of particles, but **Huygens**, publishing a paper in 1673 , saw light as a wave. In 1803 English Thomas **Young** we met before, revived the wave theory of light by performing his famous experiment called "double-slit experiment" still studied in high school, and established that monochromatic light behave as a wave, showing the fabulous strips patterns. French Augustin **Fresnel**$(1788-1827)$ also made many experiences(like interference of polarized light)to strengthen the wave nature of light. He showed that the vibrations of the light wave are transverse to the line of propagation.

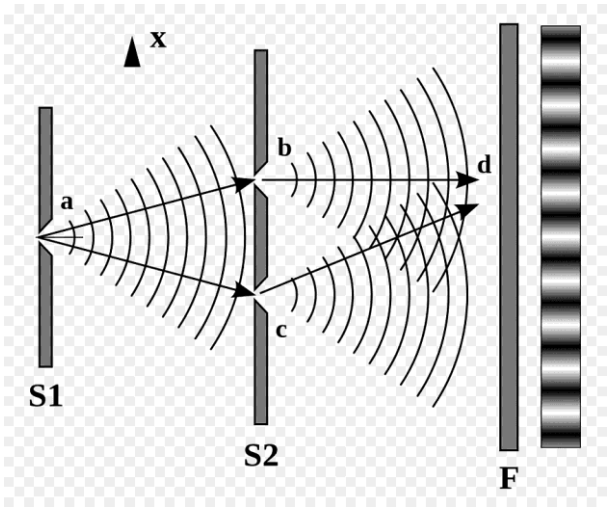


Fig. Young experiment

Light is a ***transverse*** wave. A transverse wave has an ***angular velocity*** $\omega = \lambda\nu$, ω is also called the angular frequency. The positive number λ is the ***wavelength*** and ν the frequency of the wave, $\nu = \frac{1}{t_p}$, t_p is the ***period***. The equation modeling waves was found by French Jean **D'Alembert** in a form of a partial differential equation (PDE) of the form

$$u_{tt} = v^2 u_{xx}$$

a wave of a vibrating string is characterized by its ***position*** u at the point x and at a time t so the solution of the above EDP is of the form $u(x,t)$. The equation $u_{xx} - v^2 u_{tt} = 0$ is usually noted :

$$\square u = 0$$

$\square$ being the *d'Alembert operator*. v is the speed in which the wave propagates, called phase speed in the case of light $v = c$. The equation can be solved by using **Fourier** method of separation, we suppose that the solution has the form $u(x,t) = f(x)g(t)$, then we solve two ODE, to get the solution

$$u(t) = Asin(\omega t + \phi)$$

the equation of a simple harmonic motion. The function sin is periodic of period 2π.

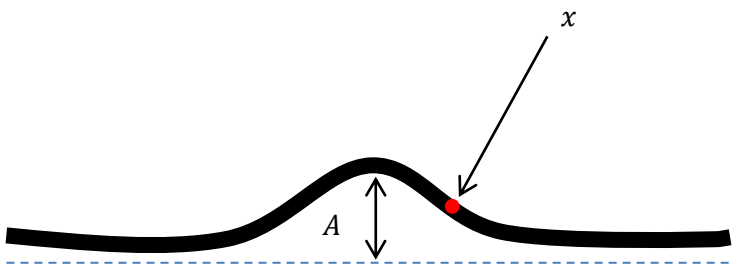


Fig. the amplitude of the rope wave

The number A is the *amplitude*, the maximal perturbation. ω the angular speed. ϕ is the *phase*. We can calculate the speed and acceleration by derivation

$$u'(t) = v = A\omega cos(\omega t + \phi)$$

Another oscillating apparatus is the *harmonic oscillator*

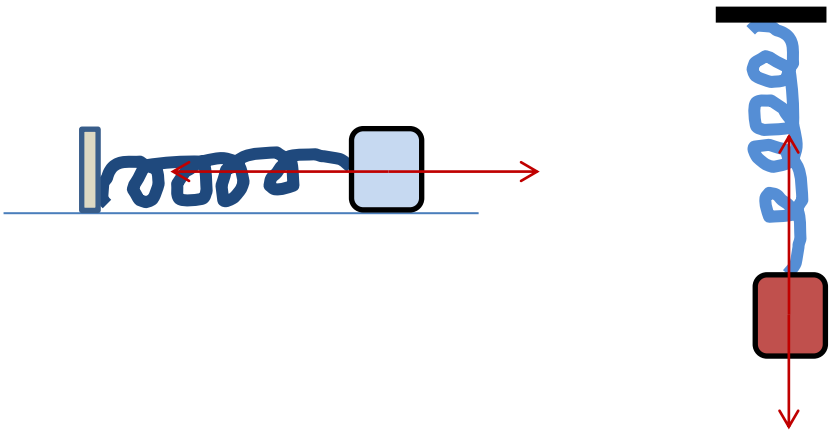


***Fig**. harmonic oscillators.*

In this case and by using dynamics law we can get an ODE of degree 2 whose solution is the same motion law as in the precedent case

$$u = Asin(\omega t + \phi)$$

Propriety of light

Light as a wave that has many properties discovered by physicists:

Refraction: when light changes propagation form a medium to another different medium(like from air to water ,from glass to air or from air to free space) the path changes direction and speed of light changes too

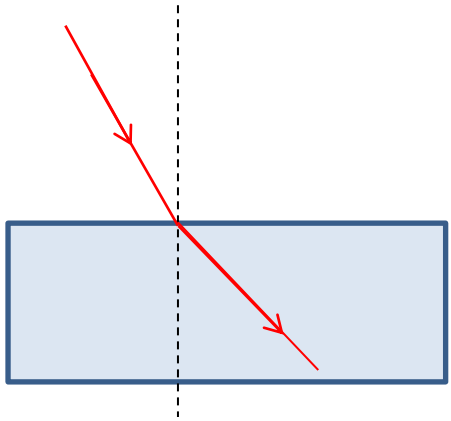


Fig. refraction : the light path changes, according to Snell-Descartes law.

According to **Snell-Descartes** law[19]

$$n_1 sin(\alpha_1) = n_2 sin(\alpha_2)$$

Diffraction: the phenomenon of diffraction of light was discovered by Italian Francesco **Grimaldi** $(1618 - 1663)$, it is the behavior of waves when they pass through an orifice or met an obstacle.

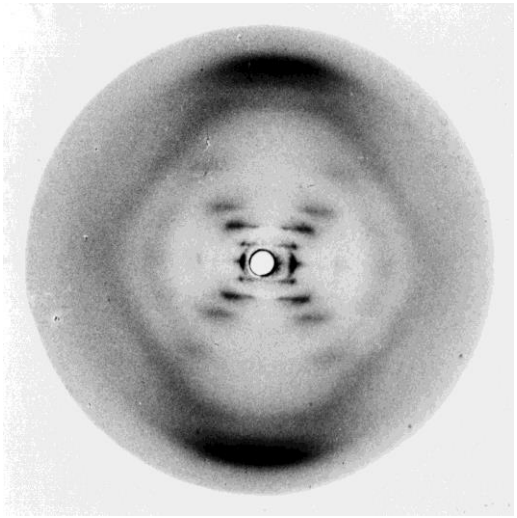


Fig. diffraction of light in the molecule of DNA.

The waves seem that they originate from the orifice itself. The diffraction phenomenon has been used as a method to discover in physics trough many historical experiments. It has also served to find out about the structure of matter like DNA. A lot of proprieties have been also confirmed by diffraction experiences like of German Max von **Laue**$(1879 - 1960)$ in his work on the diffraction of X-rays in 1912.

Polarization: another light phenomenon was discovered by French Etienne **Malus** $(1775 - 1812)$. When he was rotating a prism in a refraction experiment, he observed that some of the rays had disappeared. Light as a wave vibrates in all angles in the 3 dimension space, polarizing light means vibrating in only one angle. The early methods of polarization are due to French **Fresnel** , **Arago** and **Biot** (sound wave cannot be polarized).

[19] Discovered before them by **Ibn Sahal**$(940 - 1000)$.

Interference : it is the combination two waves together to generate a new form, like in the daily seen case of water waves. An *interferometer* is a device that shows how to inference waves and make interference pattern like **Michelson** interferometer, **Mach−Zehnder** interferometer…

Fig. Interference of waves

People were not sure if light propagated in a finite speed. The speed of light was first discovered to be finite by Danish Ole **Roemer** $(1644-1710)$ in 1676. Later many physicists used different techniques to calculate its exact value like French Hippolyte **Fizeau** $(1819-1896)$. Another French, Jean **Foucault** $(1819-1868)$, used an original technique : the revolving mirror. Their results came to a consensus on the speed to be about

$$c \approx \ 3.10^5 \ km/s$$

Some proprieties of light can be calculated:

-The *intensity* of light is the amount of power per surface.

$$I = \frac{W}{S}$$

We often says *irradiance* of light.

-The *luminosity* of light is the amount of energy radiated per unit time. The energy radiated as called also radiant power. The formula for luminosity depends on the situation.

Electricity and magnetism

A charges q forms an *electric field* around it, so at a distance r from q, the expression of the electric field is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Many different experiences on electric charges can show the lines of the field called *electric field lines*

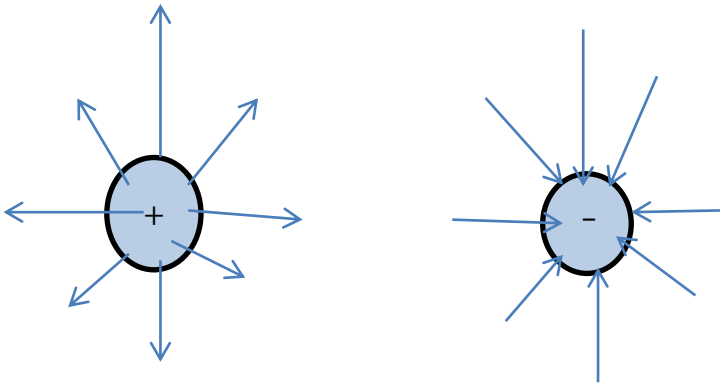


Fig. lines from a positive charge and a negative charge.

When two different charges are near enough to each other, the lines are curved and if together forming a pattern called a *dipole field*

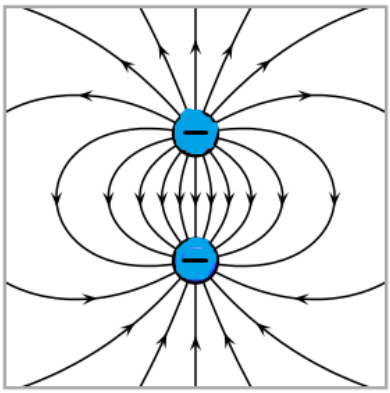


Fig. lines from two different charges

the nature of the electric field is not known, but its existence makes two sorts of forces : attraction and repulsion. If we have two charges q_1, q_2 apart, each one has a force(either

repulsive or attractive) on the other of the expression

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

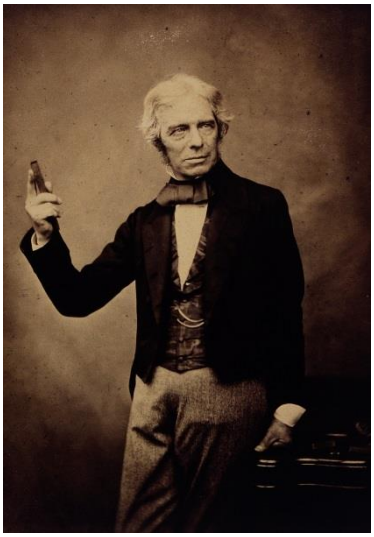
This is called Coulomb law found by French Charles **Coulomb**$(1736-1806)$ in 1784, ε_0 is the permittivity of *vacuum*, or free space. So the force exerted by a charge q_2 on another charge q_1 is $F = q_1 E$. The Coulomb law resembles Newton law of gravitation.

Magnetic force is also attractive or repulsive, yet physicists could not find the expression of the magnetic field

$$H = ?$$

it is a field associated to the magnetic force. With magnetism we discover a new concept : it is the *magnetic moment.* It generates a torque, so there is a relation between τ and H, namely

$$\tau = m \times H$$



In 1820, Danish Hans **Ørsted**$(1777-1851)$ discovered a mysterious phenomenon: when a current passes in the wires it makes a nearby light compass needle move, so obviously the current generates a magnetic field H and hence a magnetic force too. A new mysterious connection between electricity and magnetism must be elucidated. At the time, there was an English autodidact who was an assistant to chemist Humphrey **Davy**$(1778-1829)$, his name was Michael **Faraday**$(1791-1867)$. **Faraday** began to make experiences on this, and in 1821 he invented the electric motor. In France André **Ampère**$(1775-1836)$, Félix **Savart**$(1791-1841)$ and Jean **Biot**$(1774-1862)$ could even gave the law of the magnetic H

$$H = \frac{\mu_0}{4\pi}\int_C \frac{Il \times dr}{|r|^3}$$

and the value of H at distance r and current I is

$$H = \frac{\mu_0}{2\pi r} I$$

Faraday was a mediocre mathematician, but a big experimenter, in fact he made huge contributions to physics. He did not accept **Newton** view that forces acts at distance and saw that the field created by charges and magnets is a field of *lines* , *lines of* force , he was the first scientist to think in terms of *field theory*, so he viewed E and H is two fields of forces.

We still do not know what a field is. Any charge can create a field of attractive or repulsive force.
The medium for water waves is water but the medium for electromagnetic waves we do not know. So the question remained : what was the medium of waves?

The Maxwell equations

English James **Maxwell**$(1831 - 1879)$ was a prodigy, he made many contributions to physics, he invented mathematical tools to study stability, he made the first color photo... his work about the electric and magnetic fields (E, B) was decisive. He came in 1861 with 20 equations recapitulating the work done before him. By men like **Gauss**, **Ampere** and **Faraday** , about electricity and magnetism. The 20 equations have been simplified by English Oliver **Heaviside**$(1850 - 1925)$ and German Heinrich **Hertz**$(1857 - 1894)$, and the number was reduced to 4 famous equation known as *Maxwell equations*

Name	*Equation*
Gauss law of electrostatic	$\nabla \circ E = \frac{\rho}{\varepsilon_0}$
Gauss law of magnetism	$\nabla \circ H = 0$
Faraday law	$\nabla \times E = -\frac{\partial H}{\partial t}$
Ampere law of magnetism	$\nabla \times H = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t} \right)$

Fig. The four equation of electrodynamics.

In fact, before these equations, the first field equations ever written were the **Euler** equations modeling the motion of an ideal fluid

$$\begin{cases} u_t + uu_x = f(x,t) \\ \nabla \circ u = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where u is the velocity, the problem of finding solution to them equations which is still open. **Maxwell** proper contribution in the equation is the adding of the term $\mu_0 + \varepsilon_0 \frac{\partial E}{\partial t}$ in the last equation, it's called the *displacement current*, the equation says that a varying field induces a magnetic field. The $\nabla \circ$ operator containing the nabla symbol ∇ is another differential operator called the *divergence*, it evaluate the tendency of a field to go away from a point(think of a sink) or near to a point(think of a spring).

Fig. A sink($\nabla \circ < 0$) and a spring(source) ($\nabla \circ > 0$).

The symbol $\circ$ stands for the[20] scalar(or dot) product , $\nabla \times$ is still another differential operator called the *curl* or *rotational*, it evaluate the tendency of a field to rotate around a point. Note that there is a difference between $\nabla \times$ and $A \times B$.

The operators $\nabla, \nabla \circ, \nabla \times$ are here in Cartesian coordinates. We can also operate in other coordinates like : cylindrical(polar with an axis) and spherical.

For a function field $A = xi + uj + uk$, $u(x,y,z)$ and $v = ai + bj + ck$ we have

[20] There is also an *inner* product.

Two vectors	*Operators*
$A \circ B = \sum_{1}^{n} a_i b_i$	$\nabla \circ \mathrm{v} = \sum_{1}^{n} \frac{\partial u}{\partial x_i}$
$\mathrm{A} \times \mathrm{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$	$\nabla \times \mathrm{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix}$
$\frac{d}{dt}$	$\nabla \mathrm{u} = \sum_{1}^{3} \frac{\partial u}{\partial x_i} e_i$

Fig. The three differential operators.

$\rho = \frac{dq}{dv}$ is the charge density, μ_0 is the ***permeability*** of the vacuum, that magnetic field penetrates free space. $\nabla \circ \mathrm{E} = \frac{\rho}{\epsilon_0}$ Is the famous Gauss's law electrostatics

$$\frac{Q}{\epsilon_0} = \int_V E dv$$

Gauss law can find the electric flux over a surface and in special cases the electric field E itself.

$\nabla \circ$ measures the magnetic field from a point, so the equation $\nabla \circ B = 0$ means that there is no ***monopole magnet***. Similarly in fluid mechanics the above equation $\nabla \circ v = 0$, where v is the speed of the fluid, means the fluid is ***incompressible***, there is no source of fluid. There is an analogy between flow of fluid and flow of charge.

When **Maxwell** worked the four equations, he got two equations

$$E_{tt} = \frac{1}{\epsilon_0 \mu_0} E_{xx} \quad , \quad H_{tt} = \frac{1}{\epsilon_0 \mu_0} H_{xx}$$

They are two wave equations, and the speed $v^2 = \frac{1}{\epsilon_0 \mu_0}$ and thus he concluded that both

electric and magnetic are waves, so light is just the composition of an electric field and a magnetic field, it is then an electromagnetic wave, whose speed is

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \approx 3.10^5 \ km/s$$

so light is just an electromagnetic wave! A big theoretical discovery conforming the experimental!! **Faraday** had already proposed that light was a high frequency electromagnetic vibration. Unlike **Faraday**, **Maxwell** believed in the theory of ether. The fields E and H are perpendicular to each other. Until now, no one knows what an electric(magnetic) field is , such fields propagate in the vacuum, where there is nothing inside !!!

The first electromagnetic wave was detected by **Hertz** in 1888, he used what is known as *harmonic oscillator*, a linear oscillating electric dipole, it is also called a *resonator*, it is defined as : "the simplest body that can absorb and emit radiation".

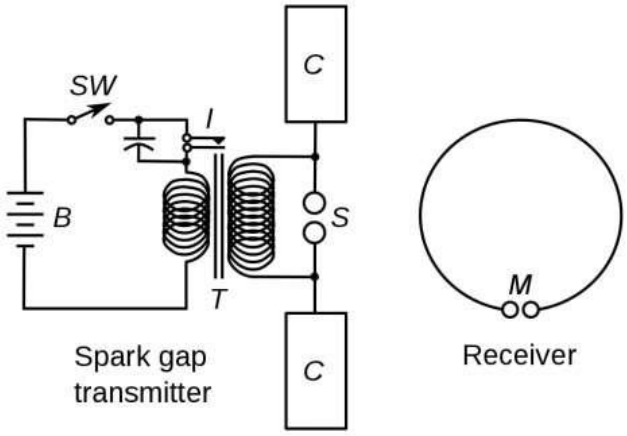


Fig. Hertz equipment

The idea behind the device is to generate varying electric and magnetic fields at the same time. We know that a static electric charge creates a static electric field and when moving it creates a magnetic field, so to make a varying field the charge must be moving and accelerating i.e. to generate an electromagnetic wave we make an oscillating charge. In fact **Hertz** generated what we call

radio waves. The waves reflect and refract as light waves confirming Maxwell theory.

It was a big achievement of the century that began modern telecommunications. So through his work, **Maxwell** made a fusion between electricity and magnetism. The discovery by **Hertz**, had freed the road to the invention of radio easy, this was made by Italian Gulielmo **Marconi**$(1874-1937)$.

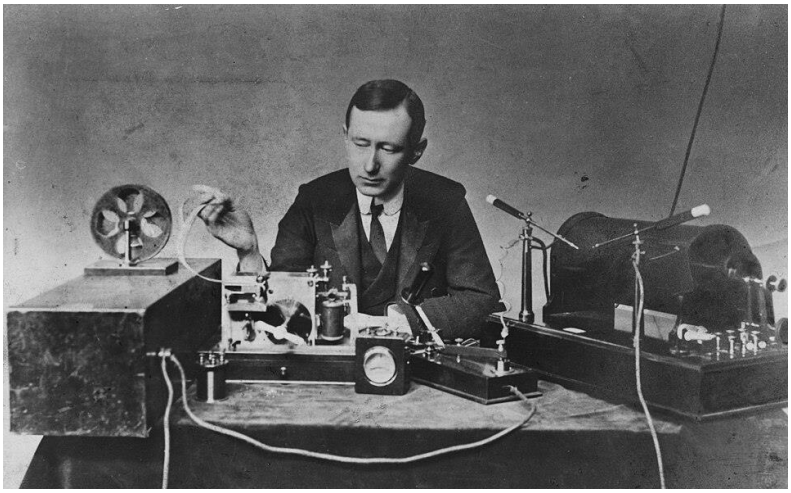


Fig. Marconi with his wireless equipment

In 1862 , **Maxwell** also claimed that an electromagnetic wave can have a momentum called *radiation pressure*, so light can make a pressure on the surface it hits. This would be proved experimentally in 1900 by Russian Pyotr **Lebedev**$(1866-1912)$. English John **Poynting**$(1852-1914)$ a student of **Maxwell** introduced the vector S called ***Poynting vector***

$$S=\frac{power}{area}=E\times H$$

it is the flow density of the electromagnetic wave through a surface in a certain direction. S is also known as the ***Umov−Poynting** vector* . He found his theorem

$$\frac{\partial e}{\partial t}+\nabla\circ S=s$$

that gives momentum $p=\frac{E}{c}$, the radiation pressure P, is given by

$$P = \frac{Force}{Area}$$

When a charge moves in a magnetic field, it is subject to a force though a law given by Dutch physicist Hendrik **Lorentz**$(1853 - 1928)$. Given a charge q of velocity ∨ moving in an electric E and magnetic H fields the *magnetic force* is

$$F = q(E + \vee H)$$

For this it is known as the electromagnetic force the *Lorentz force*. The force is exerted on a charged particle in electric and magnetic fields, it explains the interaction between radiation and matter. So the direction of a particle is changed by both electric and magnetic fields and the force is proportional to the particle speed.

The Doppler effect

In 1842, Austrian Christian **Doppler** $(1803 - 1853)$ made investigations on the color of the light arriving from stars . He observed that the color shifted to the red. Six years later **Fizeau** discovered also the phenomenon on electromagnetic waves. So sometimes the phenomenon is called **Doppler-Fizeau** effect. The effect is also observed on sound waves, When a source of sound waves moves, at the same time the waves to each other (or the opposite) and the wavelength diminishes (or the opposite), and it seems that the wave arrives to the observer in high frequency.

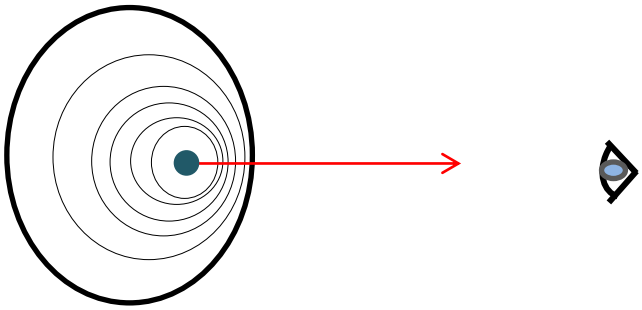


Fig. Doppler light in waves shift to red.

Fizeau noticed a difference in frequency of a wave which is moving, when the source and receptor are moving relative to each other. The same phenomenon has been discovered in the XX century in another context. When astronomers were observing galaxies, they remarked that they were emitting light that goes to a low frequency light namely to red light, a phenomenon they called *redshift*. Thy concluded that galaxies' speed was increasing, which means that the Universe was expanding. The discovery was made by many scientists including American Edwin **Hubble**$(1889-1953)$ in 1929.

Exercises

1) what is vacuum ?

1) Show that $A \circ B = B \circ A$.

1) find a such that the two vectors : $A = 2i + aj - 5k$ and $B = i - j + 4k$ are perpendicular.

1) Find the value of the angle between the two vectors $A = i + j - k$ and $B = i + j + 2k$.

1) Show that $|A \times B|^2 = |A|^2|B|^2 - |A \circ B|^2$ (Calculate each term)

1) Show that the formula $A \times B = |A||B|sin(\theta)$ is false.

1) Let the two vectors $A = -i + 2j - k$ and $B = 2i + j + k$. Calculate $A \times B$ and $B \times A$. This will show you that $A \times B = -B \times A$.

1) a body moving according to the equation

$$u = Asin(\omega t + \phi)$$

Find its speed and acceleration.

1) the period of cos is 2π. Show that the period of $tan = \frac{sin}{cos}$ is π. What about $cotan = \frac{cos}{sin}$?

1) A field F with $\nabla \times F = 0$ is called irrational. Verify if $F = xi + yj + zk$ is irrational.

1) find ∇F for the following :

i) $x + y + z$

ii) xyz

iii) $\log(xyz^2)$

1) find $\nabla \circ$, of the following vector functions

i) $xi + yj + zk$

ii) $xi + (y - z)j + zk$

iii) $(x - y)i + yj + z^2k$

1) find $\nabla \times$ of the following vector functions

i) $xi + yj + zk$

ii) $xi + (y^2 - z)j + zk$

iii) $(x - y)i + yj + z^2k$

1) When does $\nabla \circ E = 0$?

1) Let $dE = \frac{1}{4\pi r^2}dq$, integrate to find **Gauss** law for electric fields.

1) from Gauss's law for $\vec{H}$, write $\vec{H}$ in another form.

1)Show that $u^{"}(t) = -u\omega$.

1) Using the language of vector calculus, in what direction propagates the electromagnetic wave ?

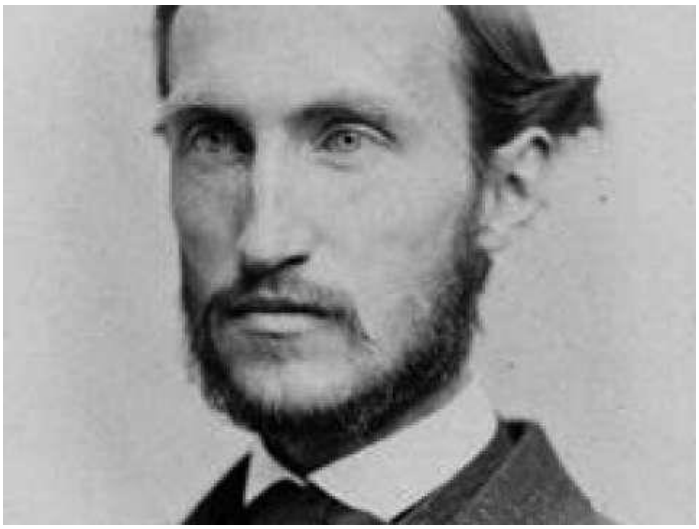
1) Show that $v = \lambda v$.

1) a station sends a wave signal in the frequency $85\ Mhz$. Find the wavelength λ.

1)is the third Newton true for thee magnetic force between two moving charges?

1)calculate the pressure of a force $F = 5\ N$ on a disk of radius $0.56\ m$.

THEORY OF GASES



Willard Gibbs did for statistical mechanics and for thermodynamics what Laplace did for celestial mechanics and Maxwell did for electrodynamics, namely made his field a well-high finished theoretical structure.

Robert **Millikan**

A gas is an example of a ***fluid***, fluids constitute a special matter that is different from solids, the other fluids next to gases are liquids. Unlike a liquid, a gas has many good characteristics like the two important properties of compressibility and dilatation. A fluid also as a matter is studied in the same manner as a rigid body , so we have : fluid statics, fluid dynamics and fluid kinematics.

Water and ***air*** are the standard fluids and most on earth. From them we can define relative density for gases and liquids. The ***density*** is defined as:

$$\rho = \frac{mass}{volume} = \frac{m}{V}$$

to get the ***relative density*** for a liquid : $\frac{\rho_{liquid}}{\rho_{water}}$, the relative density of water is 1. For gases , the relative density for a gas : $\frac{\rho_{gas}}{\rho_{air}}$, the relative density of air is 1.

One of the men that made much experiments on gases was French Antoine **Lavoisier**$(1743 - 1794)$, considered by many as the father of modern chemistry. He identified two gases : oxygen and hydrogen. Gases were studied by chemists, some were isolated as early as the $XVIII$ century as hydrogen, oxygen,…. Since the early XIX century ,

gases were currently used in in industry and turned to have many important daily practical applications.

The three parameters

Before that, scientists since the $XVII$ century have been very busy to understand the relation between three factors related to any gas, namely:

$$V, \qquad P, \qquad T$$

volume, pressure and temperature. V is the amount of space that a gas can occupy. Pressure is defined as $\frac{force}{surface}$, and measured by atm (atmosphere) and the atmospheric pressure, to which we all are subject, is defined to be $1\ atm$, which is equivalent to $760\ mm$ of mercury. T is different from heat, and any scale of temperature can do the job.

It has been shown that the three V, P, T are not independent, so the idea is this : we try to fix one of them and look for the proprieties of the two others, then find the relations

$$V(P,T), \qquad P(V,T), \qquad T(P,V)$$

This methodology has resulted in 3 laws called the *approximate laws of compressibility and expansion* :

– The *Charles law* (or *law of volumes*) due to French Jacques **Charles**$(1746-1823)$, it says that for a constant pressure P we have the relation

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} = \text{const}$$

So $V \propto T$.

– For a *constant* temperature T, we have according to the known **Boyle –Mariotte law** the relation

$$P_1V_1 = P_2V_2 = \text{const}$$

A law due to English Robert **Boyle**$(1627-1691)$ and French priest Edmé **Mariote**$(1620-1684)$. The law means that at a constant temperature : $PV = constant$, and $\frac{P_1}{V_2} = \frac{P_2}{V_1}$.
We can plot the relation $PV = constant$ and get a curve known to physicists as the *isothermal curve*[21] , that depicts the relation existing between V and P at constant temperature.

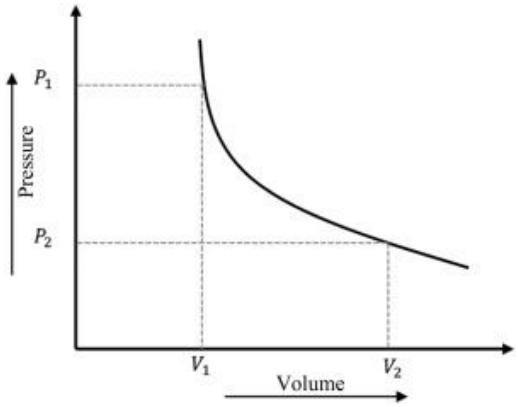


Fig. isothermal curve.

– Another law that has been established by many and named after **Gay Lussac**$(1778-1850)$ and it is answering the question : For a given constant volume V, what is the relation between P and T? It has been found that for any gas, we have:

$$\frac{P_1}{T_1} = \frac{P_2}{T_2} = \text{const}$$

for example, it has been noticed that when P diminishes, the boiling temperature T of water is less than $100\ ^oC$. This law is known also as Amontons law, after French Guillaume **Amontons**$(1663-1705)$.

Now, we can combine the three laws and composed laws

[21] From Greek *ισος* meaning "equal" , in this context "equal temperature". We will see iso again.

$$\frac{V_1 P_1}{T_1} = \frac{V_2 P_2}{T_2} = \text{const}$$

A law meaning that : $PV \propto T$. The temperatures are in the **Kelvin** absolute scale.

Ideal gases

Around the time when **Gay-Lussac** published his law, English john **Dalton**$(1766 - 1844)$ was studying the mixture components in different gases and emitted his *law of proportions*.

If two elements combine with each other to for more than one compound, the ratios of the masses of one element that combines with the fixed mass of the other are whole numbers.

Afterwards and in 1808 he advanced the theory that gases, matter,… are composed of atoms and molecules[22].

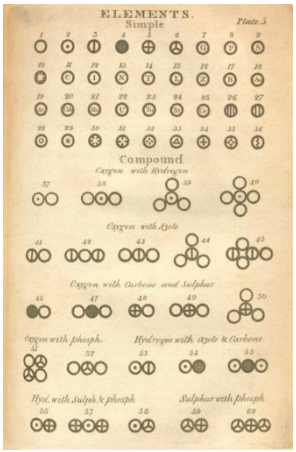


Fig. Atoms of elements according to **Dalton** from his book
A New system of Chemical Philosophy $(1808 - 1827)$

In a sense, the pressure of a gas can be considered as the percussions of atoms and molecules of the gas. In 1811 Italian Amedeo **Avogadro**$(1776 - 1856)$ proposed that the

22 See for a historical review of atomic theory the book of English physicist James **Jeans**$(1877 - 1946)$: *An Introduction to the Kinetic Theory of Gases* (1940).

volume V of any gas is proportional to the number of molecules inside it , namely : $V \propto n$. So from this law after **Avogadro law**

$$\frac{V_1}{n_1} = \frac{V_2}{n_2} = \cdots = const.$$

where $n_1,\ n_2$ are the numbers of molecules. That means that equal volumes of a given gas contain the same number of molecules of gases. To determine the size of atoms and their masses scientists had to:

−determine the mass ratios of different elements.

−the area of form the spread of a drop of oil floating on water.

−measure the rate at which two different gases mix.

At specific set of conditions called the ***normal conditions*** corresponding to the temperature $20\ ^{o}C$ and pressure 1atm. Under these conditions 22.4 liter of a gas contains the ***Avogadro number*** of molecules, that is

$$N_A = 6.02214076 \bullet 10^{23} \text{ molecules}$$

a universal number that was first determined by German Johann **Loschmidt**$(1821 - 1895)$and French Jean **Perrin**$(1870 - 1942)$. It is the definition of the ***mole*** of matter.

The mole of matter (gas, liquid,…)is the quantity of matter contained in the Avogadro number of molecules under the standard condition of pressure and temperature.

After people began to accept the idea that a gas is composed of atoms, they began to study and understand gas motion. Of course , the situation was difficult for the huge number of

molecules. So to simplify the study physicists usually suppose two helping hypotheses:

- The random motion of molecules.
- The particles do not interact.

So this leads us to the idea of *Ideal gas*. An ideal gas is different from a real gas in that we discard some real proprieties to facilitate the uses of mathematics tools to understand the law of gas motion. Their efforts resulted to a new law for them. So **Gay Lussac** law for *ideal* gases[23] at low pressure is

$$\frac{PV}{T} = nR$$

n is the number of *moles*, R is the ideal gas constant $R = 8.314\ J/mol\ K$, it is the known *ideal gas law.* It turns out that all four laws seen before can be reduced to the unique ideal gas law. We can speculate about the above identity saying that PV is energy(force multiplied by volume) as the temperature T itself is related to energy($PV = aT$). This corresponds to Joule second law that internal energy of an ideal gas depends on temperature only, and not on volume and pressure

$$U \sim PV = f(T)$$

A pressure P was set as standard, it is the *atmospheric pressure*.

Kinetic theory of gases

The later discussion belong to *kinetic theory of gases* , one of the parts of thermodynamics, it proposes to study a gas and sees it as composed of a big numbers of particles. We have already made two suppositions on a gas to its dynamics, but we will develop them further. Statistics of movement of particles and molecules is so interesting and its importance can be hardly exaggerated, because it has been used all the time along the history of physics including quantum mechanics. The idea that a gas is composed of particles moving was made by Daniel **Bernoulli**$(1700 - 1782)$ around 1738 in his book *Hydrodynamica*. He was

[23] For real gases physicists use the empirical equation $\left(P + \frac{an^2}{V^2}\right)\left(\frac{V}{n} - b\right) = RT$, due to Dutch Johannes van der **Walls**$(1837 - 1923)$, who proposed it in 1873.

the first to begin kinetic theory, he explained that heat is caused of the motion of particles of the gas. Unfortunately his ideas were not taken seriously. In the middle of the XIX century physicists 1850 German **Clausius** said :

…to this it must be added that other facts have lately become known which support the view, that heat is not a substance, but consists in a motion of the least parts of bodies. If this view is correct, it is admissible to apply to heat the general mechanical principle that a motion may be transformed into work, and in such a manner that the loss of vis viva is proportional to the work accomplished.

the philosophy at the time was statistics, so a gas is seen a cloud of" particles" (molecules or monatomic gas like neon) and as one cannot understand the movement of one particle, so to understand their movement as a whole we use the method of statistics (we will see later in quantum theory similar method: Bose statistics, Fermi statistics).

The owner of the theory faced serious problems.

–The speed of a particle inside a gas is big. Physicists used the *average* of speed.

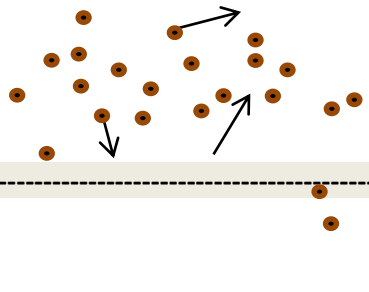
– with collision and with the problem of energy it is inadequate to apply Newton laws(in analytic mechanics).

–The density of a gas is $\frac{M}{V}$, the velocities of particles are to be considered ,

–the collision are to be elastic thus conserving kinetic energy.

But this study is impossible without other, they proposed that the theory tried to explain how the particles moved giving adequate postulates:

- Gas is composed of a big number of molecules of one atom or more. They are of small size.
- the molecules move in empty space and are subject to Newton laws of motion, so we can predict the motion of particles.
- the collisions of molecules are elastic, do not turn the motion more complex and difficult.
- the interactions between molecules are negligible. If not the motion is more complicated.
- kinetic energy of the molecules are proportional to : KT.



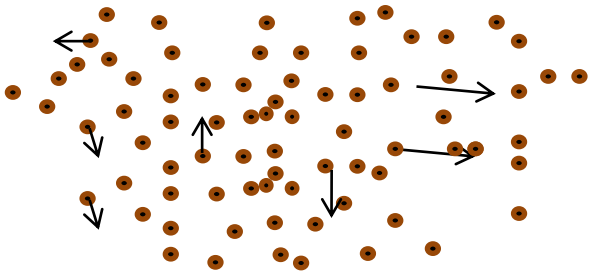


Fig. Molecules of gas in motion

English **Maxwell** made important work on the kinetic theory of gases. He was inspired by the work of **Clausius**, he studied the velocities of the particles. He thought in terms of *probabilistic* prevision of the movement[24], a big novelty in physics. He discovered that the law that modeling the velocities is governed by the normal[25] distribution probability. According to this, the number of particles between x and dx is

$$N\frac{1}{\alpha^3\sqrt{\pi}}e^{-\frac{v^2}{\alpha^2}}dx$$

and the number of particle whose velocity lies between v and $v+dv$ is

$$N\frac{1}{\alpha^3\sqrt{\pi}}v^2e^{-\frac{v^2}{\alpha^2}}dv$$

Another contribution came from German Ludwig **Boltzmann**$(1844-1906)$, who did important work and emitted his known equation gases dynamics. If $f(x,t,v)$ is the distribution function depending on position, time and speed we have that

$$\frac{\partial f}{\partial t}+v\nabla_x=Q(f,f)$$

[24] In his paper : *illustrations of the dynamical theory of gases*(1860).

[25] Called also the **Gauss-Laplace** distribution of probability. It is defined as : $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. So it is the Maxwell speed distribution.

and the velocities of the particles follow the ***Maxwell–Boltzmann distribution*** .

Both **Maxwell** and **Boltzmann** believed in atoms, **Boltzmann** suffered from a big opposition in his country[26] by physicists like **Mach**, and in Germany like Ernest **Zermelo**$(1871-1953)$. If we look at the previous law

$$\frac{PV}{T} = nR$$

In the kinetic theory of gases some law using the **Boltzmann** constant, number with mole $N_A = \frac{N}{n}$ and $\frac{R}{N_A} = k_B$, k_B the Boltzmann constant, so

$$nR = Nk_B$$

the law becomes

$$PV = Nk_BT$$

Another term depending on the temperature

$$K = \frac{3}{2}k_BT$$

So from the previous equation there is no difference from as PV is energy. So we have obtained an astonishing derivation:

Temperature is the mean kinetic energy of particles of the gas.

[26] Some say this this opposition compelled him to commit suicide.

It was Scottish William **Rankine**$(1822-1888)$ that introduced the term *potential energy* in 1853, and **Kelvin** coined the term *kinetic energy* through their work in thermodynamics.

American Josiah **Gibbs**$(1839-1903)$ did important work in vector calculus and theory of gases. He coined the term *statistical mechanics*, through his work in kinetic theory of gases in his book *Elementary Principles in Statistical Mechanics*(1902). Statistics mechanics is the part that tries to explain the behavior of matter from the motion of microscopic particles. It has a strong link to thermodynamics. It can be said that it is the statistical approach to laws of thermodynamics.

Kinetic gases theory is just a part of the statistical mechanics. Statistical mechanics is the use of statistical methods like probability to study the dynamics of a lot of small particles and to their motion. **Gibbs** worked on the significance of the equation

$$\frac{\partial \rho}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial \rho}{\partial q_i}\dot{q}_\iota+\frac{\partial \rho}{\partial p_i}\dot{p}_\iota\right)=0$$

where ρ is the probability to find in a given phase at the time t. This is called ***Liouville theorem*** , in statistical mechanics and **Gibbs** saw it as a fundamental equation.

There is also the problem of irreversibility, scientists argue in this way: since the laws of mechanics are reversible, new different postulates must be postulated to encompass the irreversibility of heat change in the science of thermodynamics.

that **Maxwell** demon in 1867 we will see with the second law of thermodynamics.

The absolute zero

Temperature T is measured by a thermometer as we will see, people could not measure heat. There are two different scale of measuring the temperature.

–The first was **Celsius** scale to choose the freezing of water as 0^o so -10 is under zero.

–The second is **Fahrenheit** that freezing is at the $32\ F$.

The two scale are both still in use[27]. So the question[28] is :

Can the temperature decreases under $0\ ^0C$ *without limit* ?

This is really a logical question because in the **Gay-Lussac** law $\frac{P}{T} = const$, if we decrease the pressure P , the temperature T also decreases so there is a lower bound for temperature as P approaches 0 , as it cannot be negative.

Chemists also have understood that there is a limit to the negative degrees. Among the persons who studied the physical case when there is no heat at all, was English Lord **Kelvin** $(1824 - 1907)$. The idea was not new at the time. In 1848 kelvin coined the *absolute zero* of temperature :

Is there any principal on which an absolute thermometric scale can be founded ? It appears to me that Carnot's theory of the motive power of heat enables us to give an affirmative answer….the characteristic propriety of the scale which I now propose is, that all degrees have the same value; that is that a unit of heat descending from a body A *at the temperature* T *of this scale, to a body B at the temperature* $(T-1)^o$*, would give out the same mechanical effect, whatever be the number* T*. This may justly be termed an absolute scale, since its characteristic is quite independent of the physical properties of any specific substance.*

By calculations he found that in the Kelvin degree in Celsius centigrade

$$1K = -273.15\ \ ^oC$$

and new scale the kelvin degree is born which is used in thermodynamics, when there is no heat at all , atoms are at rest not moving so without kinetic energy. So there is not negative kelvin degree. Relation between Celsius and Kelvin

$$K = C + 273.15$$

[27] In particular, the USA use the Fahrenheit scale of temperature.

[28] Many persons before **Kelvin** thought of a lower limit of heat: the first was Guillaume **Amontons**, then others like : George **Martin**, Robert **Boyle**, Johan **Lambert**, **Lavoisier**, **Laplace**, **Dalton**…

but

$$1K = 1\ ^{o}C$$

At the absolute zero degree, or the zero Kelvin, the particles stop moving. This degree of temperature has never been reached in laboratory. The temperature in outer space is very cold, it is about 2.7 kelvin. So the matter can be :

solid, liquid , gas

There is also another concept called the *triple point*. A triple point where all matter : gas, liquid and solid coexist at certain pressure and temperature. The triple point of water correspond to $T = 273.15$ and $P = 0.0006036\ atm.$

This raises the question of gases *liquefaction* : can we liquefy gases? We know that gases at room temperature(about $20\ ^{o}C$) are vapor[29]. As water liquid becomes a vapor at $100\ ^{o}C$, in the same manner other gases as vapors can become liquids at some temperature. The difference is the temperature at which the evaporation takes place. Around 1844, **Faraday** has succeeded to liquefy a lot of gases, like : CO_2 , chlorine, cyanogen, ammonia,… but some important gases resisted the procedure.

So the question is : how can we liquefy gases? There are some techniques:

- To decreases pressure, when gas is pressed, its temperature drops. The pressure depends on temperature. So there is a critical temperature above which we cannot liquefy a gas by pressure. so to liquefy the gas, one has to press bellow the critical temperature.

- To use substances of like sulfurus, carbonic acid, …with pressure.

- To increase its volume, so the temperature drops when a gas expands into vacuum, this is the Joule-Kelvin effect discovered in 1852.

[29] Here we mention that matter has three states(or phases): solid, liquid, gas and plasma.

This latter method was used in France by French Louis **Cailletet**$(1832-1913)$ and in Germany by German Carl von **Linde**$(1842-1934)$ and in Britain by English William **Hampson**$(1854-1926)$, all three arrived at liquefy some gases. An important discovery was that the ebullition temperature of a liquid depends on the pressure. Similarly the temperature of water ebullition is not always $100\ C^o$, it changes with altitude, which means by changing pressure.

Gas	*boiling point C*	*Date*
water	$+100$	1742
oxygen	-183	1833
air	-194	1877
nitrogen	-195	1882
hydrogen	-253	1898
helium	-269	1908

Fig. Temperature at which the gas becomes a liquid, called also *"boiling point".*

The air itself as a gas (composed of different gases) can be liquefied. The resulted liquid is called *liquid air* , the operation was performed in 1877. In 1908 , Dutch Heike **Onnes**$(1853-1926)$ succeeded to liquefy the difficult helium gas. In 1869 Thomas **Andrews**$(1813-1885)$ discovered the *critical temperature* , it is the temperature at which we cannot liquefy the gas by pressure. So every gas has its critical temperature, above which it is impossible to transform it to a liquid.

It is by this idea that the refrigerator is constructed.

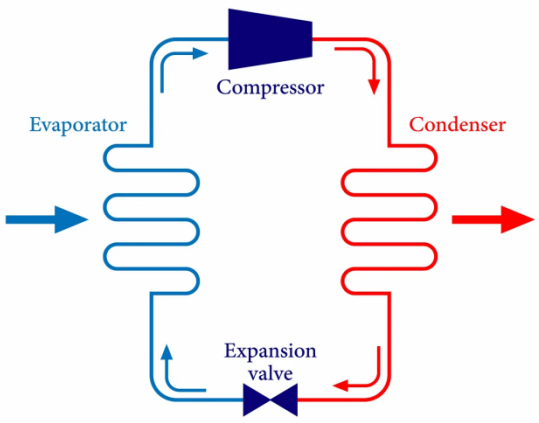


Fig. refrigerator cycle

A liquid(called a refrigerant like : Freon, *tetrafluoroethane*) is filled inside the refrigerator through a valve and expanses, becomes a gas of low pressure(becomes a vapor), so it takes all the heat from inside the refrigerator, then it is pushed by pressure to outside to the compressor to become a liquid anew, so it is compressed to a high pressure, it leaves the heat gained to the outside, and again to the valve and the cycle continues.

Exercises

1) Oil floats on water because is less dense. The mass of oil for volume. What is a specific density of oil?

1) we have a recipient of helium of volume 1L of 200 K, what would its volume et 250 K?

1) we have oxygen of volume 1L at the pressure 2 atm, what would its volume at 4 atm?

1) we have $1L$ of oxygen of pressure 2.5 atm of 200 K, what would its temperature at $4\ atm$?

1) we have of volume of 22 L at temperature 300 K and. What would be its volume at the normal pressure at temperature 300 K.

1) show tha the laws Boyle ,….can be derived from ideal gas law.

1)using perfect gas law, find the volume of 2 mole of the gas at the standard conditions.

1)How many particles are there of helium in 2 mole of the gas ?

1)How many moles are there of helium in 2.10^{26} molecules of the gas O_2 ?

1) Transform in Celsius degree the following

$$12K, \qquad 100K, \qquad 273K$$

1)put in Fahrenheit degree the following

$$1K, \qquad 10K, \qquad 299K$$

1) what temperature in Celsius is 100 Kelvin?

1)The water becomes a vapor at the Celsius degree $100\ ^oC$ what corresponds in Kelvin?

1) at what temperature in Kelvin do both centigrade and Fahrenheit scale have the same numerical value?

1) how many moles are there in 2g of helium?

THE CARNOT CYCLE

The caloric developed in the furnace by the effect of the combustion travers the walls of the boiler, produces steam, and in some way incorporates itself it. The latter carrying it away, takes it first into the cylinder, where it performs some function, and from thence into the condenser, where it is liquefied by contact with the cold water which encounters there.

Sadi **Carnot**

Thermodynamics is an important part of physics, and one of the physical subjects that helped to begin quantum theory. The Greek word *therme* $(\theta\varepsilon\rho\mu\eta)$ means heat, so the dynamics of heat is the science that tries to understand and explain the heat phenomenon: its nature, how it propagates,…. Although there were a lot of confusions about the real nature of heat, physicists came to understand heat Q as a form of energy

$$Q \sim E$$

One of the principal task of thermodynamics is to find out the relation between heat(energy E)and temperature T.

People succeeded first to measure temperature T, the device to do this work is called a *thermometer* , developed by many scientists by using liquids like : water, alcohol,… until German Daniel **Fahrenheit**$(1686-1736)$ replaced liquids with mercury inventing his proper scale F . Swedish Anders **Celsius**$(1701-1744)$ followed the task with very clever

idea; he chose 0 for the point of freezing of water and the 100 as the boiling point, so there are exactly 100 degrees between 0 and 100, it is the *centigrade* scale of temperature or the *Celsius scale* C. And the relation between **Celsius** and **Fahrenheit** scales is

$$C = \frac{5}{9}(\mathrm{F} - 32)$$

Although people could measure temperature T, they could not measure heat, there was a confusion between the two concepts. So question haunted people since long: What is temperature T ? what is heat Q? what is the relation between T and Q ?

Heat

One of the early thoughts about heat are those of English John **Locke**$(1632 - 1704)$, who said in 1720

Heat is the very brisk agitation of the insensible parts of the objects, which produces in us that sensation, from whence we denominate the object hot; so what in our sensation in heat, in the object is nothing but motion[30].

Sun is the principal source of heat. Heat also can be generated during many processes like chemical reactions,…. It can also be generated by mechanic means like friction. It has many proprieties :

–it may be transferred by conduction, convection and radiation.

–Goes from hot to cool.

–It can cause gases and metal to expand.

Heat also can be a *radiant heat*. it can be generated from the passing of current in metals because of the resistance of metal through the known **Ohm law** due to German Georg **Ohm**$(1789 - 1754)$, his law is given by the relation $R = \frac{V}{I}$, The heat power(energy per unit time) emitted by this was found by James **Joule**$(1818 - 1889)$

[30] In his "*Collection of several pieces…*" p. 224.

$$E = RI^2$$

known as *Joule law*. To get bigger amount of heat we have to raise the value of the resistance R, to get to that we use special metals[31] with big resistance R. When hot meets with cold, we get a *thermal equilibrium*, with the same temperature.

As early as the $XVIII$ century, physicists raised many questions about the nature of heat. there is for example a relation between heat, mass ad temperature. In 1762 , English Joseph **Black**$(1728 - 1799)$ introduced :

—The important concept of *latent heat*

—Specific heat, which is the amount of heat needed to raise the temperature of mass m from T_2 to T_2 and its value is given by expression : $c = \frac{Q}{m(T_2 - T_1)}$.

French Antoine **Lavoisier**$(1743 - 1794)$ was one of the early chemists that advanced the idea that heat was a fluid, which he called" an igneous fluid" later known under the name *caloric*. He says :

We know, in general, that all bodies in nature are immersed in caloric, that they are surrounded by it, penetrated in all sides, that it fills all the spaces left between their molecules. In certain cases, caloric fixes itself in bodies, even in such a way as to constitute their solid parts; but that most often it separates the molecules, it exerts a repulsive force on them, and that it is by its action or its greater or lesser accumulation that depends the passage of bodies from the solid state to the liquid state, or from the liquid state to the air state[32].

This hypothesis of the caloric looks like the "ether" hypothesis that plays the role of a carrier of the electromagnetic waves in the electromagnetic theory. In 1822 French Joseph **Fourier**$(1768 - 1830)$ published his book *Analytic theory of Heat*, introducing the partial differential equation that models heat diffusion in a body

$$u_t = \alpha u_{xx}$$

[31] For example tungsten as a natural metal, has a high resistance. To get higher resistance we usually use *alloys*.

[32] In "*Elementary Treatise of Chemistry* " (1789).

x can be a vector in space, and α is positive number presenting *thermal diffusivity*. The equation can be seen as a relation between heat Q and temperature T:

$$\begin{cases} \nabla \circ Q = -c\dfrac{\partial T}{\partial t} \\ Q = -k\nabla T \end{cases}$$

where c is the specific heat and k the *thermic conductivity*. We have already seen the wave equation $u_{tt} = v^2 u_{xx}$, and with the Laplace equation $\Delta u = 0$, they are the three important linear PDE in mathematical physics

$$\begin{cases} u_t = \alpha u_{xx} \\ u_{tt} = v^2 u_{xx} \\ \Delta u = 0 \end{cases}$$

the equation $\Delta u = 0$ has many application that we will see. The value of heat solution at the point x at a given time t is

$$\frac{1}{\sqrt{4kt\pi}} \int_{\mathbb{R}} e^{\frac{-(x-z)^2}{4kt}} f(z)dz$$

where $f(z)$ is the value of heat at $t = 0$. In his book, **Fourier** introduced the celebrated *Fourier series* much used in physics since then, it has the general form

$$\sum_{n=0}^{+\infty} a_n cos(nx) + b_n sin(nx)$$

the series is supposed to represent a function in an interval(because trigonometric functions are periodic). **Fourier** left many things to be proved about his series. But Fourier series has gained much use in modern mathematics and many branches of physics like quantum mechanics.

By physical considerations, the argument (nx) is sometimes written $(n\omega_0 x)$ where ω_0 is the *fundamental frequency* and nx can be. A Fourier series can have a complex form

$$\sum_{-+\infty}^{+\infty} c_n e^{in\omega_0 x}$$

where c_n are complex numbers. Fourier series belongs to domain of mathematics called *hormonic analysis*. We have two : there is **Fourier** analysis that given a function f as a Fourier series, so we look for (a_n, b_n) and **Fourier** synthesis such that given coefficients (a_n, b_n) we construct the function f.

Steam engine

Putting aside Fourier series and heat diffusion, it is known that heat can be *transformed to work* W, and work can also be turned into heat.

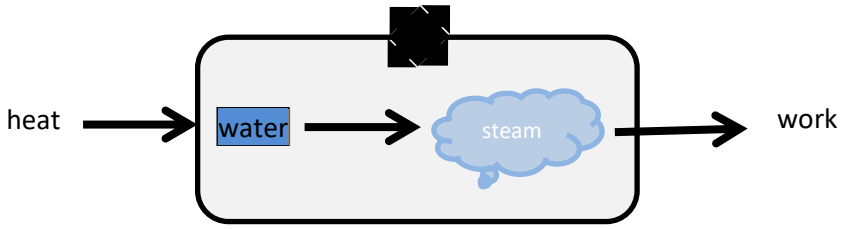


Fig. The idea of the steam engine.

This fact was used by engineers like English James **Watt**$(1736 - 1819)$ to develop the *steam engine* around 1770 after the invention of Thomas **Newcomen**$(1664 - 1729)$ of the *beam engine*. The heat is by burning material(coal, wood) , then a *latent heat* that given to change to liquid to steam in our case ,then the use of steam pressure a motion that generates power.

.

Fig. A pumpjack works by the same principle to extract oil.

When heat turns water to steam, the steam is used as a power to move the piston. Steam push the piston to rotate the forth and back movement and piston it a close the valve to steam and when it return it open it to let steam in the cylinder. Many improvements have been added like *valves*. The role of a valve is to control the flow of steam. When the steam goes from the cylinder to the condenser via the valve, and this by the of the piston there is a waste of heat in the process. So the problem is how to compensate that waste of power.

Watt has made some perfections in the steam engine :

- the separate condenser.
- parallel motion.

The word “watt” becomes the unity of power.

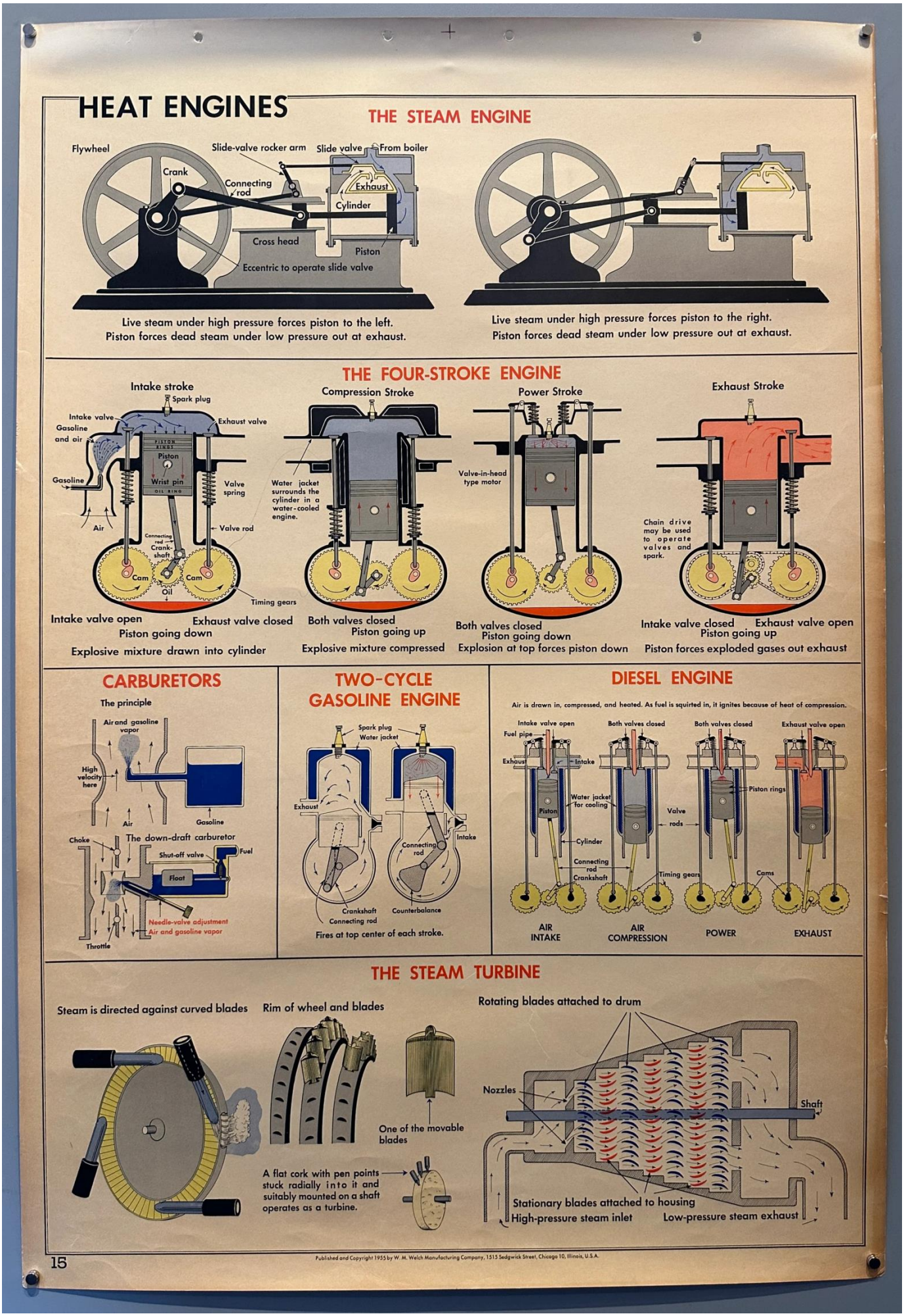


Fig. Mechanical structure of a simple steam engine.

The invention of this steam engine was not only to extract water from ground for mining industry, but also to the locomotion of new engines. It was used to move the first locomotive in 1804 by English Richard **Trevithick**$(1771-1833)$.

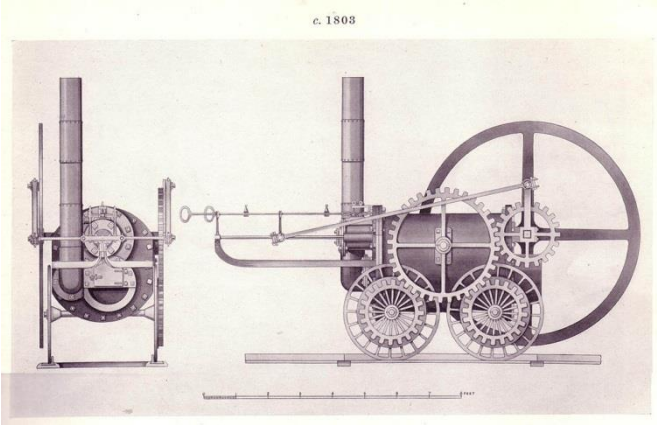


Fig. The Trevithick locomotive

Trevithick once, around 1799, brought improvements to the boiler of steam engines to give more power to the steam(strong steam), he later added safety valves.

Heat and work

French Sadi **Carnot**$(1796-1832)$ exposed his ideas on engines like the steam engine in his essay entitled "*Reflexions sur la puissance motrice du feu…*" of the year 1824

Everyone knows that heat can produce motion. That it possesses vast motive-power no one can doubt, in these days when the steam-engine is everywhere so well known

His best thoughts turned around the problem of the *efficiency* of an engine, he imagined an *ideal steam engine*, so his question was : under what conditions is the performance of a steam engine *optimal* ?

The question has often been raised whether the motive power of heat is unbounded, whether the possible improvements in steam engines have an assignable limit-a limit which the nature of things will not allow to be passed by any means whatever; or whether, on the contrary, these improvements may be carried on indefinitely.

He even speculated about the materials used in the engine to maximize the performance of the engine

We have sought, and are seeking today, to ascertain whether there are in existence agents preferable to the vapor of water for developing the motive power of heat; whether atmospheric air, for example, would not present in this respect great advantages.

He proposed

—The use of a liquid having a higher point of ebullition.

—there is a big loss of motive power due to the drop(dissipative) of temperature between the fire(source of heat) and the water.

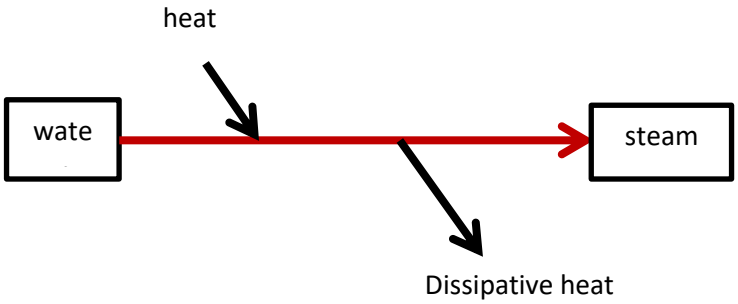


Fig. heat and steam

so there is a *dissipative* heat which one has to minimize. So the balance between produced work and lost heat

$$water + heat = steam + dissipative\ heat$$

So **Carnot** imagined to construct a hypothetical engine using an ideal gas(steam) that has an optimal *efficiency*. But two :

– Isothermal change: under the same temperature so $\Delta T = T_2 - T_1 = 0$

– Adiabatic change : without transferring heat between the two systems. No real system is adiabatic but it is only an approximation. So isothermal is because if we want to maintain a constant T we have to exchange heat the two systems.

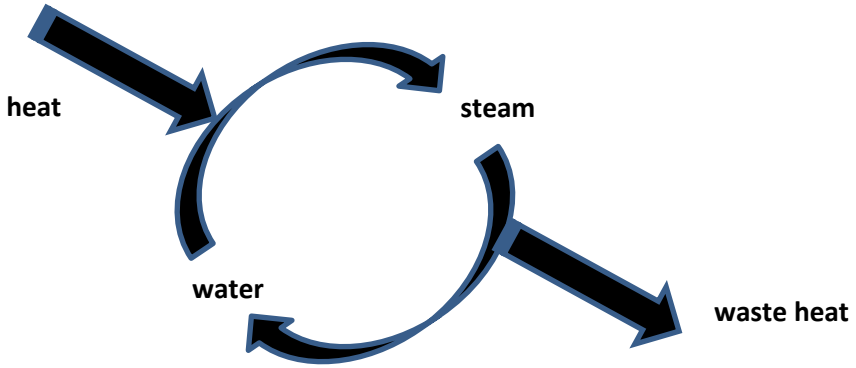


Fig. A water-steam cycle

He argues that *agents* (like water, air) provide the same work(by the way, he talked in terms of the caloric!). So the efficiency e of the engine is

$$e = \frac{work\ done}{heat\ given}$$

or and that the production of mechanical work during a cycle comes from transmission of heat from higher to a lower temperature. He meant that the efficiency of an engine depends on the temperature only. He gives an analogy :

We can compare with sufficient accuracy the motive power of heat to that of a waterfall. Each has a maximum that we cannot exceed; whatever may be, on the one hand, the machine which is acted upon by the heat. The motive power of a waterfall depends on its height and on the quantity of the liquid; the motive power of heat depends also on the quantity of caloric used, and on what maybe termed, on what in fact we will call, the height of its fall, that is to say, the difference of temperature of the bodies between which the exchange of caloric is made.

So there is no engine that has the same efficiency for all temperatures.

$$e = \frac{T_H - T_L}{T_H} = 1 - \frac{T_L}{T_H}$$

So

$$work\ done\ < geat\ given$$

Efficiency is like probability never exceeds 1, in fact unlike probability it never reaches 1.

The Carnot cycle

Efficiency problem led **Carnot**, to his hypothetical engine known also as *Carnot cycle* .

When a body has undergone any changes and after a certain number of transformations is brought back identically to its original state, considered relatively to density, temperature and mode of aggregation, it must contain the same quantity of heat as it contained at first.

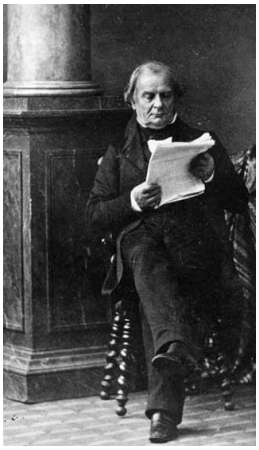
He declares that the of an engine can be in a set of cycles or operations :

- isothermal expansion : at same temperature
- adiabatic expansion : without exchange of heat.
- isothermal compression : at same temperature
- adiabatic compression : without exchange of heat.

So the next result of **Carnot** for efficiency was the introduction of the concept of reversibility,

The most efficient engine is the one in which all the operations are reversible.

we know now that the processes of thermodynamic are irreversible. It is highly improbable that heat goes from cold to hot.



After the death of Carnot, another French that in this subject was Émile **Clapeyron**$(1799-1864)$ in a paper published in 1834 "*Sur la puissance motrice de la chaleur* ", he discussed and expended Carnot ideas and made known within scientific community even outside France.

Clapeyron introduced also his graph representing a **Carnot** cycle. The cycle pressure P in terms of volume V:

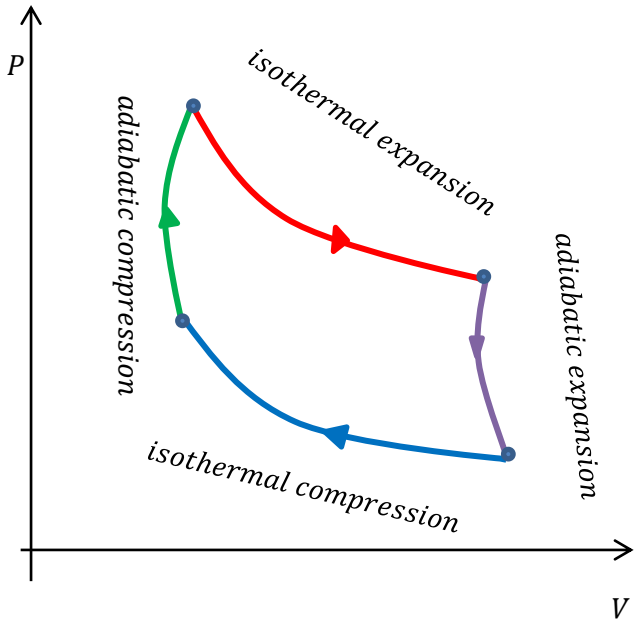


Fig. Carnot cycle

So the above graphic is like a function, and **Clapeyron** looked for a function of time $f(t)$ that **Carnot** called *Carnot's function*. **Thomson** known as Lord **Kelvin**, tried without success to find the function

$$f(t,T) =?$$

Carnot made some mistakes in his reasoning but the ideas expressed in the book would be very influential. Afterwards they freed the way to physicists to found modern

thermodynamics.

Exercises

1) what is the temperature where Celsius and Fahrenheit are the same ?

1) calculate the heat energy for current of $I = 0.5\ amp$ and $R = 2\ Ohm$.

1) wrote the first term of a **Fourier** series**.**

1) find the expression $\int \cos(nx)\,dx$ and $\int \sin(nx)\,dx$.

1) find the value of $\int_0^{2\pi} \cos(nx)\,dx$ and $\int_{-\pi}^{\pi} \cos(nx)\,dx$. What do you remark ?

1) show that the **Fourier** series is periodic.

1) what is form of **Fourier** series when f is odd and when f is even ? prove it.

1) Find Δu for the following functions:

i) $u(x) = 2x - x^3$.

ii) $u(x, y) = x - z^2 y$.

iii) $u(x, y, z) = x^4 - y + z^2$.

iii) $u(x, y, z) = x^4 - y + z^2$.

1) show that $\Delta u = 0$ for the following functions:

i) $u(x) = x - x^3$.

ii) $u(x, y) = x - zy$.

iii) $u(x, y, z) = x - y + z^2$.

NUMBERS AND MATRICES

Talk to Hermite! He never evokes a concrete image, yet you soon perceive that the more abstract entities are to him like living creatures.

*Henri **Poincaré***

Quantum physics has used a lot of classical mathematical tools. Then before entering into its machinery, in what follows we need to remind the reader of some of them, these include abstract algebra like linear algebra (vectors, matrices,…) and group theory, Another related concept is group. Groups are very abstract objects that have seen useful in theoretical physics since the beginning of the last century.

And we take also some glimpses of complex analysis with complex numbers. Complex analysis is an autonomous branch of mathematic, and it is used in the domain of differential equations and geometry. It was first introduced in some physical studies like electricity and will have a dominant role in founding many concepts of quantum theory.
We have seen already the concept of vector and saw some examples of physical entities that can be presented by vectors like : forces, … we add here that between two vectors v_1 and v_2 we can construct a new vector called *direct sum*(or sometimes direct product)

$$v_1 \oplus v_2$$

in the same manner between two vector spaces V_1 and V_2 we can find a new vector space of the same dimension called the direct sum

$$V_1 \oplus V_2$$

we will see matrices and later other operations on them. We will discover an important concept that generalizes vectors and matrices: the concept of *tensor*. Tensors are extensively used in theoretical physics.

Complex numbers

Complex numbers were born in suspicion and there was time when they were not considered as numbers at all, they were called *imaginary* numbers. They were first introduced as solutions of polynomial (or algebraic) equations like $x^2 + 1 = 0$. A complex number is a number of he form

$$z = \mathrm{x} + iy$$

where x and y are real ($\in \mathbb{R}$) . Complex numbers verify all the operations that real number verify, they also constitute a *field*, it is the field of complex numbers noted $\mathbb{C}$, and we have $\mathbb{R} \subset \mathbb{C}$. A *conjugate* of z is $\bar{z} = x - iy$. Complex numbers are the tool of *complex analysis* , a very important branch in mathematics in general. After number theory and geometry, complex numbers find their way even in physics in the matrices of **Heisenberg** , in **Schrödinger** equation,…. One of the central formulas in complex analysis is the equation

$$e^{ix} = \cos(x) + isin(x)$$

found for the first time by **Euler**. It gives $e^{i\pi} = -1$ one of the most amazing formulas in mathematics. The number e^{-ix} is the conjugate of e^{x}. And the norm (the positive distance from the origin(of a complex number is

$$\|z\| = |z| = z \bullet \bar{z} = \sqrt{x^2 + y^2}$$

Linearity

Linearity is a very desired propriety in mathematics. It is for example a good propriety for physical laws, linearity makes the things simpler. Many equations we will encounter are

linear. Unfortunately, almost all nature laws are nonlinear. So **Newton** second law is not linear because $V(x)$ is not linear in general. In the case of quantum , we will met different situations, **Schrödinger** equation is linear, whereas **Yang-Milles** equation is nonlinear…
We say that a function f is *linear* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Where α, β are real numbers or complex in general. Among linear operators we know : derivation $\frac{d}{dx}$ and integration $\int$.

A *linear* (or *vector*) space is a set of vectors. A linear space E is something just like $\mathbb{R}^3$, the space where we do geometry. $\mathbb{R}^3$ is an infinite collection of points which we call vectors. An important vector space in the quantum theory is the **Hilbert** space. What is interesting about linear spaces is that we can form linear functions between them called *linear maps* or *linear operators*

$$f : E \longrightarrow F$$

The dimension of a linear space is linked to the number of linearly independent vectors in it. The base(or basis) in a linear space of vectors is a of vectors $(e_1, \dots, e_n)$such that every and vector v

$$v = \sum_{i=1}^{i=n} a_i e_i$$

The linear space $\mathbb{R}^3$ has the dimension 3. For example, any n axes that are *perpendicular* constitute a *basis*. A vector can be written in many basis and so the basis is not unique.

The magnitude of a vector v is its *norm* , the positive distance from the origin, for a vector $v\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ we have :

$$\|v\| = \sqrt{a^2 + b^2 + c^2}$$

A vector whose magnitude is 1 is the *unit vector*. We can the unit vector v by dividing it by the magnitude. An *orthonormal* system of vectors is composed of unit orthogonal vectors.

Matrices

Matrices come from linear systems. A linear system $AX = Y$ transforms a vector $X(x_1, \dots, x_n)$ to another vector $Y(y_1, \dots, y_n)$. The coefficients of this transformation are what constitutes an array called a *matrix*. The coefficients a_{ij} of a matrix can be real or complex numbers. So the rows (or columns) of the matrix are the law of transformation or precisely a linear transformation. A matrix can do many known transformations :

- stretching.
- reflection.
- rotation.
- shear…

We will encounter many interesting transformations like the **Lorenz** transformation, **Poincaré** transformation,…

In modern mathematics language, matrices are seen as *objects* like numbers, functions,…A matrix of order nm is a matrix of n rows and m columns. A matrix of nn is a matrix of n rows and n columns, it is a *squared* matrix of *order n*. Squared matrices are very important in applications, they can be of finite order like the 2×2 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

or of infinite order like

$$\begin{bmatrix} a_{11} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{bmatrix}$$

the latter has an infinite number of rows or columns. We can find in physics a complex matrix

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

We can perform between matrices operations well defined as

$$AB\,, \qquad A+B, \qquad A-B, \qquad \alpha A$$

The operations $A+B,\ A-B,\ \alpha A$ commute, but the product of matrices may be not commutative[33] so $AB \neq BA$.

The *trace* of A noted Tr, is the sum $\sum\ a_{ii}$ the diagonal coefficients.
A transpose of a matrix A is A^T that is $a_{ij} = a_{ji}$ (the rows become columns and the columns become rows). A matrix A is *symmetric* if $A^T = A$. If we take the transpose and the conjugates of numbers of a matrix A we get the conjugate transpose $\bar{A}^T$ also noted A^*. A matrix is *Hermitian*[34] or self-adjoint if

$$\bar{A}^T = A$$

Every real symmetric matrix is Hermitian.

A known result in linear algebra says : to every matrix corresponds a linear map and vice versa[35]. A matrix of order nm is a function from a linear space of dimension m to another linear space of dimension n.

An inverse of a matrix is a matrix A^{-1} such that $AA^{-1} = I_n$. Where is the identity matrix, when $n = 2$ it is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix that has an inverse is *nonsingular* , otherwise it is *singular*.

A matrix is *orthogonal* if

---

[33] There is a system of numbers called quaternions found by William **Hamilton**$(1805-1865)$ of the form $r = a + bi + cj + dk$ where $ij = jk = ik = -1$ and the product is not commutative in general : $r_1 r_2 \neq r_2 r_1$.

[34] After French mathematician Charles **Hermite**$(1822-1801)$ who studied them.

[35] When the space E is an infinite dimensional space, a linear map may be not presented by a matrix.

$$A^{-1} = A^T$$

The orthogonal matrices have for determinant one of the umbers

$$-1, \qquad 1$$

There are also unitary matrices. A ***unitary*** matrix A is a matrix whose inverse is its conjugate transpose is called ***unitary*** , so

$$A^{-1} = A^* = \bar{A}^T$$

Every real orthogonal matrix is unitary. The unitary matrices have for determinant one of the umbers

$$-1, \qquad 1, \qquad i, \qquad -i$$

We have until now

$$\begin{aligned} symmetric &\leftrightarrow hermitian \\ orthogonal &\leftrightarrow unitay \end{aligned}$$

An interesting propriety of squared matrices is their ***eigenvalues***. We say that the number λ is an ***eigenvalue*** for A if it verifies the equation $Ax = \lambda x$. To find the eigenvalues of a matrix A we solve the equation

$$det(A - \lambda I) = 0$$

Some matrices have no eigenvalues. The product of the eigenvalues of a matrix A is its determinant, and the sum is the trace of A.
For every eigenvalue there is ***eigenvector*** $Ax = \lambda x,\ x$ is the ***eigenvector*** associated with the eigenvalue λ. A theorem due to **Hermite** says that all eigenvalues of symmetric matrices are real. We have seen that the inverse of a matrix A is A^{-1} such that $A\,A^{-1} = I$.

The eigenvalues of matrices play an important role in quantum mechanics, exactly through eigenvalue problem

$$Ax = \lambda x$$

A is a linear operator and it is very important to find the eigenvalues and eigenvectors. Many equations in physics are eigenvalue problems like the **Sturm-Liouville** equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y = -\lambda w(x)y$$

In the place of the matrix A there is derivation which is a linear operation $\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$. The equation is an ODE of order 2. The task is to find the eigenvalue λ for which there is a solution.

Diagonalization

When the dimension of the linear space is finite, a known result in this domain is that every matrix represents or is associated to a linear map and to every map corresponds a matrix. One of the goals of linear algebra is how to solve linear systems. This especially necessary , when the number n is huge. So we try to reduce(reduction of matrices) the matrix to an easy form.

One situation that facilitates this task is when the matrix A is *diagonalizable*, that is when A is equivalent to a diagonal matrix D

$$A = P^{-1}DP$$

There are some conditions if verified, the matrix A can be diagonalizable :

– the dimension of its eigenspaces is n.

– the eigenvalues of A are different.

Any real symmetric or Hermitian matrix is diagonalizable.

Group theory

The theory of abstract algebra, including linear algebra, is a branch whose tools are very used in mathematics and in modern physics. In fact, modern physics uses concepts from abstract algebra that are seemingly very far from physics, like groups. A group is a set of elements with an algebraic structure inside it. More precisely, a *group* is a set of element with a law " $*$ " such that

- $\forall a, b \, \exists c : \; a * b = c$
- $a * (b * c) = (a * b) * c$
- $\exists e : a * e = a$

The group is noted $(G,*)$. A group is abelian[36](or commutative) if $a * b = b * a$ for all elements a and b, otherwise it is non Abelian(or non-commutative). Group theory began with French Evariste **Galois**$(1811 - 1832)$ around 1831 in relation with the solution of algebraic equations. Afterwards, the theory has been developed to an autonomous subject inside what is called abstract algebra. Groups as structures, can be classified in a lot of different types

finite groups, quotient groups, symmetric groups , cyclic groups, *simple groups, normal groups, solvable groups, semigroups, action groups, discrete groups, topological groups,…*

Groups , as we will see often in what follows, show and describes a very important propriety in nature : symmetry.

A matrix is considered as a transformation. A collection of matrices can form a group under multiplication or addition, there are then many interesting groups of matrices:

- The linear group $GL(n)$, GL stands for general linear group, the group of all invertible matrices of order n.
- The orthogonal group $O(n)$, the group of all orthogonal matrices of order n.
- Another group is the special orthogonal group $SO(n)$, group of matrices of order n with determinant 1.
- The unitary group $U(n)$, the group of all unitary matrices of order n.

[36] In honor of Norwegian Niels **Abel**$(18 - 18)$ who played a role in related concepts in abstract algebra.

–The special unitary group $SU(n)$, the group of all unitary matrices of order n an determinant 1.

And we have

$$SO(n) \subset O(n) \subset GL(n)$$
$$SU(n) \subset U(n) \subset GL(n)$$

One has to notice that the group law $*$ is a kind of ***transformation*** , it transforms an element to another. So we can consider that the whole group can be seen as a transformation of a set to another set, this transformation is called an ***action*** , it is the action group on the elements of a set.

An important group is the ***Lie group***[37] which we will see later. There are also in the domain of relativity many groups : **Galilei** group, **Lorentz** group, **Poincaré** group,…

Exercises

1) write the each linear transformation in a form of a matrix

i) $y_1 = 5x_1 + x_2, \quad y_2 = x_1 + 3x_2$
ii) $y_1 = 5x_1, \quad y_2 = -x_1 + x_2$
iii) $y_1 = x_1 + x_2, \quad y_2 = 4x_1 + x_2$

1) Find the conjugate of :

i) $1 - 3i$
ii) $-i$
ii) $\frac{5}{i}$

1) Say if the following functions are linear
i) $f(x) = x + 2$
ii) $f(x, y) = x + y$
iii) $f(x, y) = \frac{x+y}{x}$

1) show that coefficients of a vector in a basis are unique.

1) show that $e^{-ix} = \cos(x) - isin(x)$. What is $\overline{e^{ix}}$?

[37] By Norwegian mathematician Sophus **Lie**$(1842 - 1899)$.

1) find the unit vector u for the vector: $v\begin{pmatrix}2\\2\\3\end{pmatrix}$.

1) let $A=\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}$. Calculate A^2-A.

1) let f a linear map. show that $f(0)=0$.

1) find the transpose conjugate of the following matrices

$$\begin{bmatrix}1 & 3 & i\\1 & 2 & 2\end{bmatrix},\qquad \begin{bmatrix}2 & i\\3 & 1\\1 & 1\end{bmatrix}$$

are they Hermitian ? why?

1) show that very real symmetric matrix is Hermitian.

1) find the eigenvalue of the following matrices

$$\begin{bmatrix}1 & 3\\1 & 2\end{bmatrix},\qquad \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix},\qquad \begin{bmatrix}i & 0\\0 & i\end{bmatrix},\qquad \begin{bmatrix}-i & 0\\0 & -i\end{bmatrix}$$

1) Let $a\neq 0$ find the eigenvalue and eigenvectors of the following matrix

$$\begin{bmatrix}a & 0 & 0\\0 & a & 0\\0 & 0 & a\end{bmatrix}$$

1) Find the traces and of every matrix

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},\qquad \begin{bmatrix}0 & -i\\i & 0\end{bmatrix},\qquad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

1) Calculate the product

$$\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix},\qquad \begin{bmatrix}1 & -5\\2 & 1\end{bmatrix}\begin{bmatrix}i & i\\0 & i\end{bmatrix},\qquad \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}-i & 1\\0 & -i\end{bmatrix}$$

1) find the identity matrices for : $\begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\end{bmatrix}$

1) find the identity matrices for : $\begin{bmatrix}2 & 3\\2 & 3\\6 & 5\end{bmatrix}$

1) Show that $A=\sqrt{A\circ A}$.

1)Find the unit vector orthogonal to $A(2,1,-1)$ and $B(0,1,1)$.

1) show that if λ is an eigenvalue of A , then λ^{-1} is an eigenvalue for A^{-1}.

1) say if the following are diagnosable

$$\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix},\qquad \begin{bmatrix}1 & -5\\2 & 1\end{bmatrix},\qquad \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}$$

1) show that any Hermitian matrix is diagonalizable.

1)what are eigenvalues and eigenvectors of $\begin{bmatrix} cos & sin \\ sin & cos \end{bmatrix}$?

1) Show that the group identity element e is unique.

1) Show that $\{-1,1\}$ is an abelian group under the operation " $\bullet$ ".

1) can the empty set { } (or ∅) be a group?

RELATIVITY

I cannot but regard the ether, which can be the seat of an electromagnetic field with its energy and its vibrations, as endowed with a certain degrees of substantiality however different it may be from all ordinary matter.

Hendrik **Lorentz**

The mechanics of **Newton** is founded on force, mass, space and time. The idea of relative motion has for origin the work of **Galilei** and **Newton** and their personal view of time and space. The Principle of Relativity had already mentioned by **Galilei** in his book[38] , in fact he says

The laws of physics are the same for all inertial frames of reference. Which mean that the laws are <u>invariant</u> in those frames.

The Principle of Relativity is equivalent to the equivalence principle. Space and time for classical mechanics are both ***<u>absolute</u>***. It is in this way that **Newton** thought of the two parameters of motion , and this view is called the Galilean relativity. So to understand the evolution of the idea of relativity we must review the important ingredient of any motion , space and time:

[38] See p. 105.

Absolute space : a space is where we live, we do our experiences. By absolute we mean here something that is independent of outer things. So spaces are distinct of any object, so it is known by the distances between objects. Empty space is *homogenous* : it is the same at every point and it is *isotropic* : the same in any direction.

Absolute time: the time nature is different of that of space. We can feel according to our experience if time seems flows quickly or slowly. When we are in an experience that we enjoy, we feel that time flows quickly, and it flows slowly if we are in a boring situation. Also the time for a sleeping person is different for a waken person. But for physics time is universal, the absolute time passes equably without regard to anything external and hence without reference to any change or to any way of measuring of time like hours, days, months, and years.

These have been severely criticized, but we cannot blame much the people at that time, for they followed the given data, they could not think otherwise. The problem began when physicists consider an object is accelerating or moving at the speed near to the speed of light c. So relativity theory we are going to see in what follows is called special relativity, it is a new understanding of space and time. It is different from general relativity that discusses gravity.

All began with Maxwell equations and ether. **Descartes** was perhaps the first to introduce the idea of ether. According to him, the *ether* was the medium of the force $\vec{F}$. Many physicists like French **Fresnel** also made many experience to show the wave nature of light and stressed on the ether hypothesis. Then **Maxwell** reintroduced it(1864) and thought that ether was necessary as a medium to transport electromagnetic waves. So did other scientists as **Lorentz** and **Poincaré**, they believed in the existence of the ether that must fill the space ….. many proprieties of this ether were proposed.

So the aim of physicists of the time was to prove either the existence or the non-existence of hypothetical ether. After an idea suggested first by **Maxwell**, two Americans Albert[39] **Michelson** $(1852-1931)$ and Edward **Morley** $(1838-1923)$ had an experiment in 1887 known as the Michelson-Morley experiment. By which they wanted to detect an *ether wind* by an interferometer, otherwise the aim of thee experience was to detect *the movement of the earth with respect to the ether*, for that they sent a signal (light!) to a mirror which sends it back and in different directions : in the orbital direction and the opposite direction. Their result showed that there is no time difference for different signal movements. So the consequence of this negative result is that electromagnetic waves, unlike sound waves, do not need a medium to propagate, they move in vacuum at the

[39] American **Michelson** visited German Hermann **Helmholtz**$(1821-1894)$ in 1881 and was acquainted of the interferometer designed by French Jules **Jamin**$(1818-1886)$.

speed of light.

The Galilean transformation

When a body, or more bodies, moves with respect of another body we have a *relative* movement. For two cars moving in one direction, the first at the speed $50\ k/h$ and the second at the speed $55k/h$, we can conclude that the second is moving with respect to the first at the speed $5\ k/h$. Now the two cars are moving in two opposite directions with the same speeds as before. So the speed of the two cars to each other is the same : $55 + 50 = 105\ k/h$. The classical mechanics before special relativity knows relative speed and relative acceleration.
The motion of a body can be uniform in a frame and accelerated in another frame. As an example : if an accelerating train passes along a station where a passenger is standing, and inside the train there is a another man who is going forward so there is a difference between the motion with respect the outside man and the motion with respect to the train.

We introduce *inertial frame* also called *Galilean frame* , it is a frame where the first Newton law (inertia) holds, inertia as **Newton** says:

Each body perseveres in its state of being at rest or moving uniformly straight forward, expect insofar as it is compelled to change its state by forces impressed.

An inertial frame can also be defined as the frame which is not accelerating, or rotating, it moves with a constant speed. So the problem is the existence or absence of acceleration. In fact all three Newton laws are valid in inertial frames.

This can done by analyzing two situations:

– The first frame D_1 is static and the second frame D_2 is moving with a constant speed v.

– The frame D_1 is moving with a constant v_1 and the second D_2 with another constant v_1 such that $v_2 \geq v_1$.

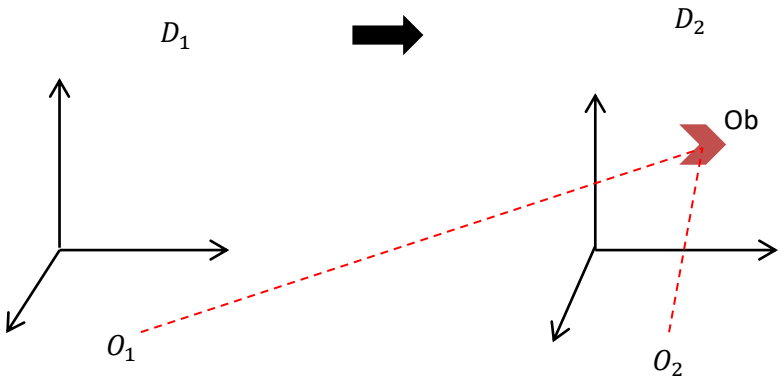


Fig. two frames

At the time $t_1 = t_2$ the two clocks measuring time in both frames are synchronized. To analyze well this situation we introduce observers. An observer O_1 in the first frame and another observer O_2 in the second frame, they observe a moving object Ob. When we evaluate the speed of the object Ob moving we get equations. These equations are named *Galilean transformations* in honor of Italian scientist **Galilei** who analyzed this motion in 1632. So in two inertial frames we can evaluate the speed as $v = \frac{x_1 - x_2}{t}$ and we have :

$$x_2 = x_1 - vt$$

$$y_2 = y_1$$

$$z_2 = z_1$$

The motion is going in one direction, he direction (x) and not in the direction of (y) and (z). The time is the same at the two places, that means the events(movements) are *simultaneous* , if we add another letter for time we have

$$t_1 = t_2$$

the time then is absolute like space $x_1 - x_2$. By derivation

$$v_2 = v_1 - v$$

And so the speed to O_1 is the sum $v_2 + v$. And by the same method we find

$$a_2 = a_1$$

The acceleration is constant. Earth is not a *Galilean frame* because it moves in the space and turns around itself. So

Galilean relativity principle applies to Newton laws

=

Newton laws are invariant under Galilean transformation.

We know that the wave equation is not invariant under Galilean transformation, does the Galilean relativity apply to Maxwell laws?

Physicist began to study Maxwell equations as a special example of the motion of a particle going with a big speed, like the speed of light c.

So physicists have proved that **Maxwell** equations are *not invariant* under the Galilean transformation.

The Lorentz transformation and new ideas

The atmosphere of the time to working physicists was to try to remain with ether theory, but they needed new ideas, physicists had to invent some helping concepts, so new original ideas began to appear in the scientific community. At the same time refutation of the ether hypothesis by the Michelson-Morley experience[40] vexed them, it showed that the speed of light is independent of the notion of the source. At the same year of the Michelson-Morley

[40] Some deny that this experience was the source of relativity.

experiment, i.e. in 1887, German Woldemar **Voigt**, showed that wave equation was not invariant under a set of transformation he invented. He also introduced the known factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

as seen from it, when the acceleration grows and the speed are near the speed of light c, the situation changes. As the factor prefigured a link to space and time :

– *contraction* for space.

– *dilation* for time.

in 1890 , in another direction, and to save the "*luminiferous ether*" theory, Irish George **Fitzgerald**$(1851-1901)$ introducing the same factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, conjectured a *length contraction* (to contract: to shorten, to be smaller) of an object in the direction of the motion so for length equation

$$l = l_0\sqrt{1-\frac{v^2}{c^2}}$$

where l_0 is the original length,

The same ideas were reintroduced by **Lorentz** in the period $1902-1905$. So $l = l_0\sqrt{1-\frac{v^2}{c^2}}$ is called the *Lorentz-Fitzgerald contraction*. And the same idea with time, so there is a *time dilation* (to dilate : to enlarge) so for time we have

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

We have seen that **Voigt** introduced a transformation like that of **Lorentz** . Both **Poincaré** and **Lorentz** worked on a new similar transformation about 1904. We have :two coordinate the first of a stationary object moving in ether, another system moving with a constant speed with respect to ether.

$$x_2 = \sqrt{1 - \frac{v^2}{c^2}}(x_1 - t_2 v)$$

$$y_2 = y_1$$

$$z_2 = z_1$$

but for time

$$t_1 = \sqrt{1 - \frac{v^2}{c^2}}\left(t_2 - \frac{x_1 v}{c^2}\right)$$

Newton laws are not invariant under **Lorentz** transformation.

Maxwell equations are invariant under **Lorentz** transformation.

The theory of relativity was then discovered by **Lorentz** and **Poincaré** whose thoughts were in the same direction. The transformation played a role in what follows :

$$x_2 = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\sqrt{1 - \frac{v^2}{c^2}}$ is a key factor and for the mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and length contraction

$$l = l_0\sqrt{1-\frac{v^2}{c^2}}$$

So two events that are simultaneous for O_1 are not necessarily simultaneous for O_2. In 1904 **Lorentz** introduced "transverse mass" and "longitudinal mass" in his paper" *electro-magnetic phenomena in a system moving with any velocity smaller than that of light*". The longitudinal mass is

$$m_L = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

And the transverse mass is

$$m_T = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

So the invariance of the speed of light is a consequence of length contraction.

The ideas of Poincaré

The other discoverer of relativity theory was French scientist Henri **Poincaré**$(1854-1912)$ in 1905 and before that date. So we see that many ideas had been already introduced by him. He stated for example the principle of *relativity* , among results conceived by him was that no object can go in the speed of light and worked with the factor

$$a = \frac{1}{\sqrt{1-\varepsilon^2}}$$

He used" Lorentz transformation" too. **Poincaré** showed that wave equation was not invariant under the Lorentz transformation**.**

$$x_2 = (x_1 - t_2 v)\sqrt{1 - \frac{v^2}{c^2}}$$

$$y_2 = y_1$$

$$z_2 = z_1$$

he showed that Maxwell equations are invariant under his transformation. **Poincaré** showed that the set of theses transformations forms a group.

Maxwell equations are invariant under the **Poincaré** group.

The new frame is composed of three coordinates (x, y, z) as usual with an extra coordinate which is the time t so we get : (x, y, z, t) , this is *spacetime*, **Poincaré** was the first to introduce this idea in 1898 , he also raised the problem of local time and the simultaneity of two events. The machinery is that **Einstein** replaced absolute space and time with space time. We will see another role of space time in general relativity.

Before **Einstein** , and in 1904 at the Saint- Louis(USA), **Poincaré** declared:

According to the Principle of Relativity, the laws of physical phenomena must be the same for a fixed observer as for an observer who has a uniform motion of translation relative to him: so that we have not, and cannot possibly have, any means of discerning whether we are or are not, carried along in such a motion … from all these results there must arise an entirely new kind of dynamics, which will be characterized above all by the rule, that no velocity can exceed the velocity of light.

in 1899 **Poincaré** found the same equation of mass-energy equivalence, namely :

$$m = \frac{E}{c^2}$$

(m is the energy content) . Austrian Friedrich **Hasenörhl**$(1874 - 1915)$ found a similar formula but of the different form: $m = \frac{4}{3}\frac{E}{c^2}$, and in 1900, German **Wien** found also the same $m = \frac{4}{3}\frac{E}{c^2}$.
Technically classical mechanics, the group for Galilean relativity is called Gallilei group and

the group for special relativity is called the **Poincaré** group.

Einstein

Is there an absolute frame of reference? Both **Poincaré** and **Lorentz** knew that Relativity theory abolished the absolute space. Another scientist that went in this subject was German Ernst **Mach**$(1838-1916)$ who influenced much **Einstein**, he criticized[41] many things introduced by **Newton**, as for example the absolute space and time. With inertial frames, he had an idea on inertia:

it does not matter if we think of the earth as turning around on its axis, or at rest while the fixed stars revolve around it. The law of inertia must be so conceived that exactly the same thing results from the second supposition as from the first

And he called it *relativity*.

In the year 1905, a year called *annus mirabilis* (miracle year), German Albert **Einstein** published[42] papers of which 4 papers were crucial. What interests us here is a paper under the title: "*On the electrodynamics of moving bodies".* The origin of this idea is a thought experiment he had previously and goes like this : he asked what would happen if he could catch up a light ray.

When we speak of relativity theory we mean by this : *relativistic mechanics* , the movement of fast-moving bodies at the speed of light. In 1905, in the paper **Einstein** emitted two postulates :

– The laws of physics are invariant (the same)for all non-accelerating observers (inertial frames).

– The speed of light c is independent of all observers.

[41] In his book *The Science of Mechanics*(1883). Before **Mach**, Irish Bishop **Berkeley**$(1685-1753)$ in his book *De Motu*(1721). His writings against the foundation of Calculus is famous as well.

[42] See *The Collected Paper of Albert Einstein* , volume II (Princeton University Press,1989).

The first postulate was not new, it is known to **Galilei** and his followers. What is new was the second on which *relativity* is founded . So the speed of light c is a constant. Newton mechanics asserts tacitly that there is no upper bound to the speed of a moving body. The mazing fact is that the speed of light is independent of the source, for example if a rocket going at speed of light and at the same time emits a signal, the speed of the signal remains c. The speed of light is completely unaffected by the movement of its source.

Unlike Newton equation, Maxwell equations are valid in special relativity because of the speed to light.

In 1905 **Einstein** announced his theory of special relativity. In the new theory *Special relativity* is " the theory of frames of reference related by the Lorentz transformation".

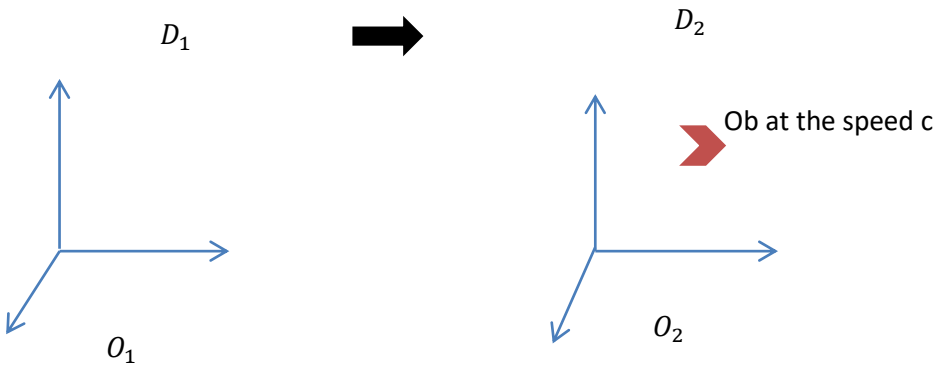


Fig. two frames

in his transformation, that m, v, t are not invariant . So Newton second law in relativity is written

$$F = \frac{d(mv)}{dt} \neq m\frac{dv}{dt}$$

We can apply the same formulas as before, for example for the dilatation formulas like speed

$$v = \frac{v_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Also the momentum is written in the form

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And kinetic energy

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2$$

Time dilation

Several authors speculated on time dilation before 1905, **Einstein** gave some speculations on the expression

$$t_2 = \frac{t_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

the main idea that stems from his is that time is not universal, an idea that would be very popular in science fiction, it explicitly says : *the time slows down* .

So two objects moving with a speed near the speed of light in motion to each other so there is no simultaneity : $t_1 \neq t_2$. So for an observer, time on a moving clock goes slower than the time in a stationary clock which is not moving. This is very well at the speed of light *the time slows down*, time passes indifferent speeds, and

$$c = \frac{contraction}{dilation}$$

So in our daily experience, a clock of an observer going at the constant speed indicating the same time different of the time a static observer, but we are not conscious of the tiny difference.

Twin paradox : Another image of time dilation is the twin travelers paradox. It is a thought experiment initiated by French physicist Paul **Langevin**$(1872 - 1946)$ in 1911. Two twins having the exact same age, the first one stay on earth and the other goes to a far planet in a speed $v \cong c$. when, after a certain sufficient time to notice difference, the second returns he will be younger than the other twin.

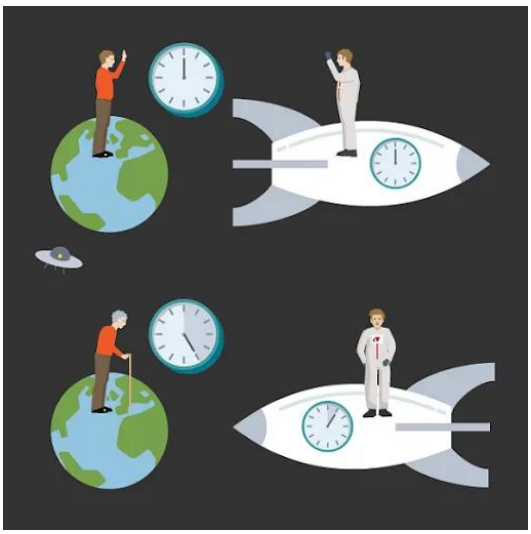


Fig. the twin paradox

Since 1932, many experiments in the spirit of **Michelson-Morley** experiments were performed to test special relativity, we can mention:

- **Kennedy-Thondike** experiments.
- **Ives-Stilwell** experiments.
- **Mössbauer** rotor experiment.

Some experiments used lasers and maser to get more precision. The experiments could also verify the time dilation. These experiments could not conducted for big bodies, they are all done to atomic particles that suit for the purpose because of the speed and size constraint,

like the particles

kaon, pion, muon

it was German Hermann **Minkowski**$(1864-1909)$, one of **Einstein**'s teacher, who gave the mathematical frame to relativity in 1908. So he proposed a system of a 4-dimensional space called the ***Minkowski space*** in which time t was the fourth coordinate . A moving body has the quadruplet (x, y, z, t) as the coordinates, the space-time in this theory is flat. We have to mention that space time of the special relativity also exist in the coming general relativity. We will see that space-time in general relativity is curved.

Mass-energy equivalence

Many scientists predicted a relation between energy and mass, some even attributed the early idea to **Newton** who said

Are not gross bodies and light convertible into another, and may not bodies receive much of their activity from the particles of light which enter their composition ?[43]

The conservation of mass was announced as a law by French chemist Antoine **Lavoisier**$(1743-1794)$ in 1789. It goes like this

The mass of an isolated chemical system remains invariable regardless of the chemical or physical transformations in which it exists.

this law of conservation was tested by Swiss Hans **Landolt**$(1831-1910)$. In 1905, the same *annus mirabilis* year, **Einstein** published another paper under the title *: "Does the*

[43] In his wok *Opticks* (1730).

inertia of a body depend upon its energy content?". In it, **Einstein** derived[44] the relation

$$E = mc^2$$

c being the speed of light.

The formula means the equivalence of energy and mass as c^2 is a constant. The relation can be illustrated by an example: the boiling of water by the action of fire (heat energy) we give to it energy E so it gets more mass and so between two energies so the

$$\Delta m = m_2 - m_1 = \frac{E_2 - E_1}{c^2}$$

Since the number c^2 is huge, the difference Δm must be very small. Another example is of a body in motion, and thus its mass changes

$$m_2 = m_1 + \frac{\Delta E}{c^2}$$

The mass-energy equivalence(mass↔ energy) by the equation $E = mc^2$ is known in many natural situations like the thermonuclear reactions in the sun, and the metabolic biology where the reactions are performed with of extrapolation of heat.

So an electron has an energy at rest noted E_0. The energy of a moving particle can be written in this theory

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The formula $E = mc^2$ is for objects that are not moving. But if $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ and $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$,

we get the known :

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44 See : the article of Jean-Paul **Auffray** : *Dual origin of* $E = mc^2$. *See also Jean* ***Hladik****: Comment le jeune et ambitieux Einstein s'est approprié la Relativité restreinte de Poincaré(Ellipses Editions, Paris-2004).*

$$E^2 = (mc^2)^2 + (pc)^2$$

It is the *energy-momentum relation* first derived by **Planck** in 1906. This equation is suitable for all situations: massless objects, big objects, moving objects at the speed of light. So when :

– $v = 0$ in $p = 0$, then $E_0 = m_0c^2$ but it was who derived it from other considerations.

– So when $m = 0$, as in the case of photon, we have $pc = E$ the energy of the particle at rest.

The photon

The physicists that discovered photoelectric effect in the XIX century could not explain it. It was not understood why electricity intensity varied in function of light frequency and not in terms of light intensity.

In the same year 1905 , Albert **Einstein** inspired from the work of **Planck** that the energy maybe quantified under the form $\varepsilon = h\nu$, he published another paper untitled : "*On a heuristic point of view concerning the production and transformation of light"*. He gave a new dynamic proposition to explain the photoelectric effect. The explanation is that the metals emit electrons under high frequency light, because that light itself is quantized and so light frequency can be measured in terms of a Planck quantum of energy

$$\nu = \frac{E}{h}$$

when a quantum of light of energy $E_\gamma = h\nu$ falls on an electron we have three energies: the first is the bound energy E_b , then the kinetic energy E_{e^-} , so the necessary energy to extract the electron is :

$$h\nu = E_b + E_{e^-}$$

Of course the electron as a body has the kinetic energy $E_{e^-} = \frac{1}{2}mv^2$. **Einstein** called $h\nu$ a *light quantum* which would be called the *photon*[45]. The supposition of **Einstein** was not

[45] The quantized sound waves are called *phonon*.

simply accepted because it has been proved by many different experiments that light was a wave, this has been proved further especially by using the newly discovered X-rays.

With the concept of photon we can explain the radiation pressure seen before and discovered by **Maxwell** in 1862. Intuitively, from classical relation $p = \sqrt{2mK}$, an object having kinetic energy K may have a momentum p. So in the new dynamics, the momentum expression becomes

$$p = \frac{h\nu}{c}$$

So the formula $E = h\nu$ proposed by **Einstein** containing the frequency ν does explain why the energy of the electrons ejected by light does not depends on it intensity and depends only on the frequency of the incident light. The photon has rest mass zero.

Exercises

1) If a bus moves at the speed $10\ m/s$, inside it a person moves forward at the speed $0.3\ m/s$. At what speed he moves to the car ?

1) if a bus moves with the speed $3\ m/s$, inside a person moves backward at the speed $0.3\ m/s$. at what speed to the car ?

1) Let the equations $r_1 = ti + (1+t)j - tk$, and $r_2 = t^2 i + j + k$ of two bodies moving. Find the relative speed and relative acceleration at the time $t = 2s$.

1) from the expression of E and p find the equation : $E^2 = (mc^2)^2 + (pc)^2$.

1) show that $E = \frac{hc}{\lambda}$.

1) If a photon has an energy of $3\ k\ eV$, find its wavelength and its wave frequency.

1) a beam of laser of frequency ν . Find the energy E of the beam in the following case :

i) $\nu = 3\ MHz$

ii) $\nu = 6\ MHz$

iii) $\nu = 9\ MHz$

what do you infer?

1) Show that is a v of a body tend to c it mass tends to ∞.

1) An electron moves at the speed of in $001\ c$, determine γ.

1) An electron moves at the speed of in $002\ c$, determine p.

1) use Lorentz formulas to calculate the mass of an electron at rest.

1) What is the distance that light travels in one year ? it is the distance unity called “light year”.

1) a body falls to the ground $g = 9.8\ m/s^2$, at what time theoretically its speed v reaches c?

THE BOHR ATOM

We are all agreed that your theory is crazy.
The question which divides us is whether it is crazy
enough to have a chance of being correct.

*Niels **Bohr***

As historians of science in general say, chance has a role in discovery, in classical physics many discoveries were made by chance like electrolysis, induction,…. hence it has even an important role in the development of science. This fact was also true for the new physical theory which is quantum mechanics, chance has played a non-negligible role. Many discoveries in this domain were mad by chance like: X-rays, radioactivity, microwaves,….

Vacuum tubes are devices developed in the XIX century. Many scientists, as German mathematician Julius **Plücker**$(1801-1868)$, observed the formation of mysterious rays emanating from the cathode $(-)$ and going to the anode $(+)$. **Plücker** discovered the cathode rays in 1858. They were named *cathode rays*. English physicist William **Crookes**$(1832-1919)$, who was also a chemist is known by constructing vacuum known called commonly Crookes tubes. Since that time of the discovery, several discoveries would take place.

Fig. Crookes tube

It was by this apparatus itself that a new type of radiation was discovered in 1895. The person who discovered it was German physicist Wilhelm **Rontgen** $(1845-1923)$, he ignored the nature of the rays and therefore he named them *the X-rays*(from the mathematical unknown x) . Afterwards , X-rays have been an essential tool in physics, especially in diffraction experiments to determine the structure of crystals, biological components…. nowadays they are used in medicine and in industry ,…

Fig. X-rays from a Crookes tube and first image of Rontgen's wife's hand.

In 1897 by using the same apparatus, English physicist Joseph **Thompson** $(1856-1940)$ discovered the first particle ever: the electron. His experience on the cathode rays consisted in the analysis of the cathode rays coming from the cathode $(-)$ going to the anode $(+)$(it was thought that electricity went from the anode $(+)$to the cathode $(-)$). So by using electric and magnetic fields, he concluded that the cathodes rays that generated X-rays are

corpuscles of negative charges, and it has been realized that electric current is just a flow of electrons from to $(-)$ to $(+)$. The word electron is from Greek $\eta\lambda\varepsilon\kappa\tau\rho\sigma\nu$, meaning amber. Ancient Greeks used to rub fur on ember to get an attractive electric force, the early use of electrostatics.

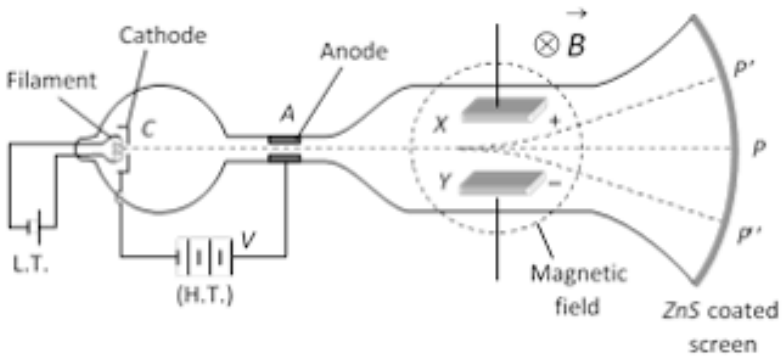


Fig. Thompson experiment

The existence of the electron was proposed as early as the 1830'. The term ***electron*** has been introduced by Irish George **Stoney**$(1826-1911)$ in 1891. **Lorentz** was the creator of the classical theory of electron. He tried to explain optics and electrodynamics in terms of the electron dynamics. because of the new objects physicists had discovered in the very small, theyintroduced a new scale of energy, it is related to the energy of small particles like the electron : it is called the ***electronvolt*** noted eV. The electronvolt it is the kinetic energy of an electron when accelerated through a voltage of 1 volt and we have following physicists,

$$1\ eV = 1.6 \bullet 10^{-19}\ Joule$$

According to the relation $E = mc^2$, the mass of electron at rest(called also rest mass or rest energy) in eV is

$$0.51 \bullet 10^{-6}\ eV$$

or

$$9.1 \bullet 10^{-31}\ kg$$

The electronvolt is a very small unit of energy, even Me(megavolt), Ge(gigavolt) and Te(teravolt) are very small compared to everyday scale of energy.

Thompson had in mind a model representing the structure of the atom with the new discovered electrons. In 1904, he proposed his *plum pudding*.

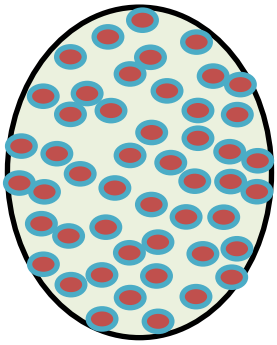


Fig. The Thompson model of the atom with negative electrons.
A Christmas pudding with plums.

He stipulated that an atom is a positively charged sphere like a pudding with the electrons on it like plums. So it not really a planetary form, but a condensed sphere where the electrons are spared on it like plums on a pudding. Not all physicists believed in the existence of atoms. In 1905, a year called *annis mirabillus* (miracle year), German Albert **Einstein** published a paper untitled :*"On the motion of small particles suspended in a stationary liquid, as required by the molecular kinetic theory of heat".* In the paper he explained the existence of atoms pointing out that the Brownian movement could be the motion of atoms. The Brownian movement is a phenomenon discovered in $1827$by Scottish botanic Robert **Brown**$(1773-1858)$.

The existence of atoms was also proved by diffraction experiments made by German Max von **Lau**$(1879-1960)$ in 1912, the technique he used also permeated to find approximately the real size of the atoms. It was only in 1979 that the atom was photographed for the first time by German Hans **Dehmelt**$(1922-2017)$,he could see the atom for the first time by trapping it in a combination of magnetic and electric fields, using a new technique known by the name *ion trap*.

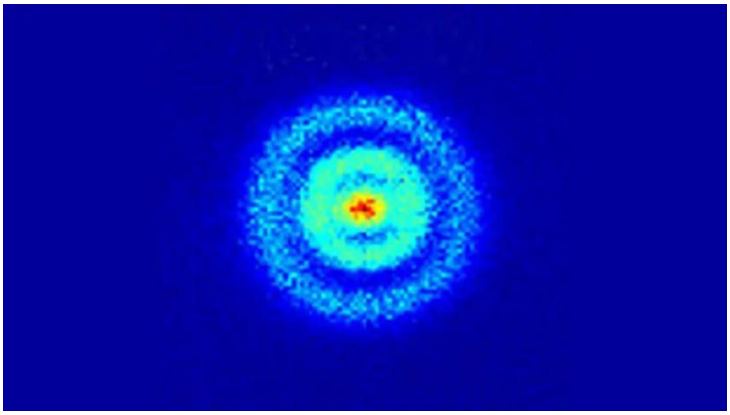


Fig. Photo of the hydrogen atom

Thomson could also measure the ratio $\frac{-e}{e_m}$, but he could not find the values of both e^- and e_m. American Robert **Millikan**$(1868-1953)$ inspired by former experiments, could measure the charge of the electron by his famous *oil drop experiment*, in 1910. The experiment consisted in creating a point of equilibrium of an oil drop using the of two forces : electric and gravitational

$$Eq = mg$$

and by using a microscope he could evaluate the droplet speed and found that the value of the electron charge was

$$-e = 1.59 \bullet 10^{-19}\ Coulomb.$$

this is called the *elementary charge*. The mass of the electron was then determined to be equal to

$$m_e = 9.1 \bullet 10^{-31}\ kg$$

Millikan could also measure the Plank constant h and found that its value was the same as found by thermodynamics considerations of the black body.

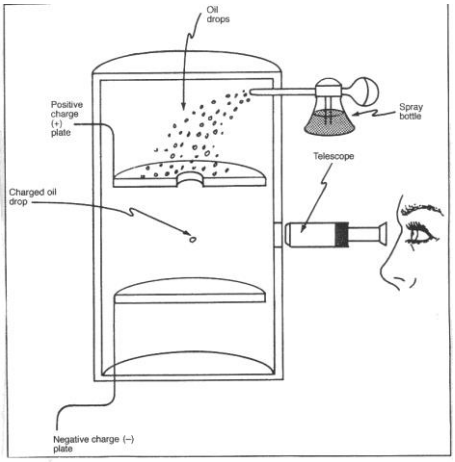


Fig. The **Millikan** oil experiment

The Rutherford experience

French Henri **Becquerel** $(1788-1878)$ was implied in the study of *photoluminescence* of elements, following the work of his grandfather Antoine **Becquerel** $(1852-1908)$. Photoluminescence is the propriety of some substances like *fluorine* to emit a mysterious florescence when exposed to sunlight. The uranium was known to be photoluminescent, it has been discovered by German Martin **Klaproth** $(1743-1817)$ in 1789 and he named it after Uranus. It was by chance, and in 1896, that **Becquerel** discovered that uranium salt emitted emanations without exposing it to sunlight. Marie **Curie**$(1867-1934)$ named the new propriety *radioactivity*. Radioactivity is the propriety, not yet explained at the time, for some elements to emit radiation spontaneously. She and her husband Pierre **Curie**$(1859-1906)$ discovered new radioactive elements: radium, polonium. She also showed that thorium(discovered in 1829) was radioactive. From uranium, physicists later extracted the *plutonium*, one of the most dangerous substances created by man.

Fig. Uraninite, plutonium and the radioactive sign

In 1895, a new student came from New Zealand to study in London under **Thompson** at the university of Cambridge, his name was Ernest **Rutherford** $(1871-1937)$. In 1899 he discovered the radon element (and be careful! : radon≠radium) . Afterwards **Rutherford** had been investigating radioactivity since 1899 with English Frederick **Soddy**$(1877-1956)$. He showed that Uranium can radiate two different types of radiation. This sort of experiments is called a *scattering* experiment. He elucidated their nature and named both of them : the first *alpha particle* and the second *beta particle*. He says first

It has not been found possible to obtain definite evidence to determine whether it be positive or negative

He then, and in 1903 , established that alpha particles was positive. The alpha particle was identified in 1907 as the helium nucleus, and beta particle as the electron. So physicists of the time were now acquainted with three new radiations

- α Radiation.
- β Radiation
- γ Radiation
- $x-ray$ Radiation

In 1909 , two of his Rutherford assistants : German Johannes **Geiger** $(1882-1945)$ and Ernest **Marsden** $(1889-1970)$ working in the laboratory, continued scattering experimentation. They made an experience using the newly discovered α −particle, the particles are emanating from the Radon atom. The experiment consisted in shooting (or bombarding) a very thin plaque of gold by the particles.

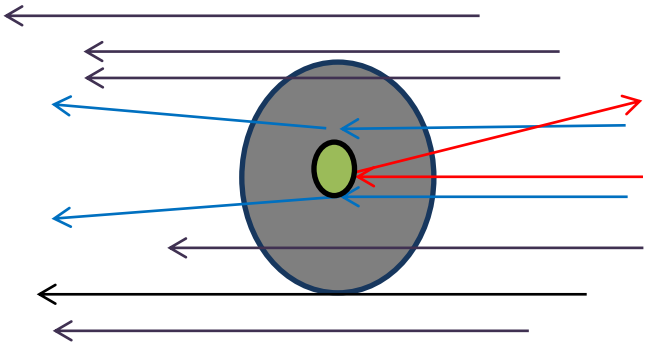


Fig. three types of particles hitting the gold atom.

They found out 3 types of particle trajectories:

– a big percentage of particles go through the plaque without deviation.

– a small percentage of particles are deviated but hit the plaque.

– a small percentage of particles go back and not penetrate.

Rutherford made a remark on the result

If the central charge be positive, it is easily seen that a positively charged mass if released from the center of a heavy atom, would acquire a great velocity in moving through the electric field. It may be possible in this way to account for the high velocity of expulsion of α particles without supposing that they are initially in rapid motion within the atom.

So **Rutherford** calculated the nucleus radius by equating the kinetic energy of the alpha particle and potential energy : $\frac{mv^2}{2} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{R}$ which gives the estimate

$$R \sim \frac{2q_1 q_2}{4\pi\epsilon_0 m v^2}$$

following these discoveries, and in 1911, **Rutherford** gave different model of the atom. He proposed his planetary[46] model for the atom

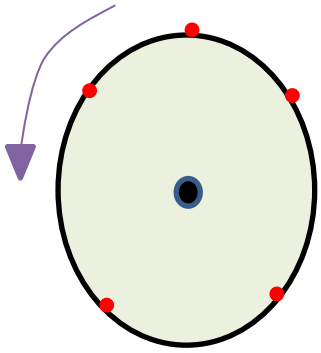


Fig. Rutherford model of the atom : the nucleus around which turn the electrons.

[46] A Japanese physicist of the name Hantano **Ngaoka** $(1865 - 1950)$ had proposed a Saturnian model in 1904.

The technique used by the two students of **Rutherford** is called *Rutherford backward scattering* , following the results he concluded that:

– The atom is composed of a center of positive charge around which orbit the electrons with negative charges.

–The atom is almost void.

–The radius of the atom is estimated to $10^{-10}\ m$.

–All the mass of the atom is situated at the center called a nucleus.

–The atom is electrically neuter.

In 1919, **Rutherford** discovered the *proton*. But what remained to physicists was to explain how this model was stable. An electron turning around the positive nucleus, according to classical physics (dynamics and electrodynamics) should have sufficient kinetic energy to maintain a stable orbit the nucleus, so two contradictory situations:

– Either it radiates and finishes at last by **Coulomb** attraction to crash on the nucleus.

– or it has a supplement kinetic energy that keeps it in its orbit around the atom. But the question arises: from where does the electron get its energy?

So the Rutherford model without other hypotheses is unstable.

The Bohr hypothesis

In 1911, Danish Niels **Bohr**$(1885-1962)$ arrived in England to study physics, first under **Thompson** then he moved to Rutherford laboratory. The serious question circulating at the time and needed solution was :

" How is the Rutherford model stable? "

This problem that faced **Bohr** would have a very different solution not known to classical mechanics. We is known at the time is that the atom is supposed to have a positive nucleon around which the electrons around it. In 1913, **Bohr** came with a completely non classical new theory. According to **Bohr**, the electron when orbiting around the nucleus, does not

radiates at tall, what happens is that the electron radiates when it changes an orbit, it either gains or loses a quantified energy E :

$$E = h\nu$$

where h is the Planck constant. It turned out that the constant h is everywhere in the dynamics of the atom. So the new postulates proposed by **Bohr** were:

–The electron can occupy only some orbits, called stationary orbits.

–The electron moves in its orbit in terms of classical physics laws.

–The electron can change orbit by receiving or emitting radiation according to the relation $E_2 - E_1 = h(\nu_2 - \nu_1)$

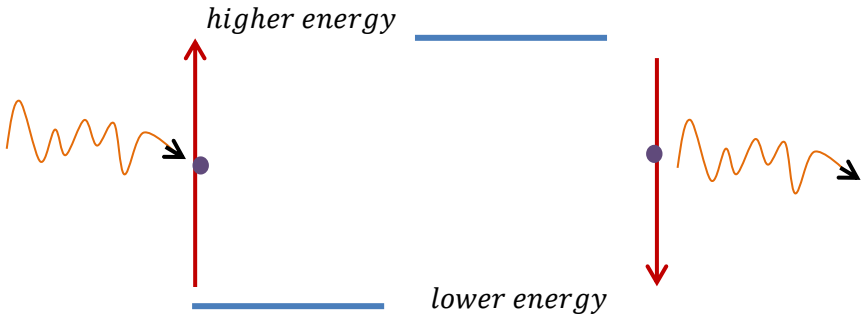


Fig. Bohr electron energy

So according to the new theory the conservation of energy is verified as usual.

The theory of **Bohr** sets on the *angular momentum*. We have seen in classical mechanics that the angular momentum is defined as

$$\mathcal{L} = r \times p$$

Observing that the **Planck** constant h has the same unit as the angular moment (energy • time), he imagined that the electrons has an angular momentum of the form

$$\mathcal{L} = m_e v r = n\frac{h}{2\pi}$$

so the angular momentum itself is quantified like energy. According to Newton dynamics, the centrifugal force $m_e \frac{v^2}{r}$ equilibrates the Coulomb force $\frac{Ze^2}{4\pi\varepsilon_0 r^2}$. So the atom is perfectly stable in this theory.

Bohr began considering his theory by applying it on the simplest atom: the hydrogen atom. Equating the two forces $\frac{Ze^2}{4\pi\varepsilon_0 r^2} = m_e \frac{v^2}{r}$ to get the equilibrium of the electron, we get the radius of the orbit

$$r = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{v^2 m_e}$$

called the ***Bohr radius***. Using his postulate $m_e v r = n\frac{h}{2\pi}$ we get

$$r = \frac{n^2 h^2 \epsilon_0}{4\pi m_e Z e^2}$$

For the hydrogen atom $Z = 1$ we have

$$r_H = \frac{n^2 h^2 \epsilon_0}{4\pi m_e e^2} = 5 \bullet 10^{-11} m$$

For the derivation of the energy $E = K + V$

$$E_n = \frac{mv^2}{2} - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}$$

In fact, the work of **Bohr** is based also on his efforts to explain rays and spectra. So his formulas should be compatible with the theory of the rays studied during the XIX century.

The spectrum of an atom corresponds to transitions made by electrons from two orbits, that means between two different levels of energy.

It is known that the theory of spectra culminated with the work of **Balmer** and **Rydberg** when they investigated the law of *spectral lines* and the frequency associated, **Balmer** gave his formula

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$$

which is a special case of **Rydberg** 's. λ is the wavelength and $\lambda\nu = c$. So **Bohr** used this to get

$$\nu = \frac{E_2 - E_1}{h} = \frac{2\pi^2 me^4}{h^3}\left(\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2}\right)$$

he found $k = \frac{1}{2\pi}$. Since h is an action, so the action is quantified and we have

$$\int pdx = n\frac{h}{2\pi}$$

called *Sommerfeld quantization condition*. The work of **Bohr** was very plausible and gave it credibility because he could by his new theory recalculate the **Rydberg** constant

$$R_c = \frac{2\pi^2 me^4}{h^3} = 109.7 \bullet 10^{-5} m$$

In 1914 the Franck-Hertz experiment done by German James **Franck**$(1882-1964)$ and Gustav **Hertz**$(1887-1975)$confirmed Bohr model by collisions of electrons with atoms. The Stern-Gerlach experiment due to German physicists Otto **Stern**$(1888-1969)$ and Walther **Gerlach**$(1889-1979)$ aimed at proving that the angular momentum was quantized confirming Bohr theory.

The mass of the atom is measured by a unit of atomic mass called the *atomic mass unit* (amu) calibrated by the carbon atom mass

$$amu = \frac{1}{12}\ mass\ of\ the\ \ C^{12}\ atom$$

It has been discovered that atoms are not static but *vibrate*. In 1967 the system international (SI) chose the second (s) to be the time of 919,263,700 vibrations of the caesium atom Cs^{55} . The first caesium clock was built in 1955 by English Louis **Essen** $(1908-1997)$.

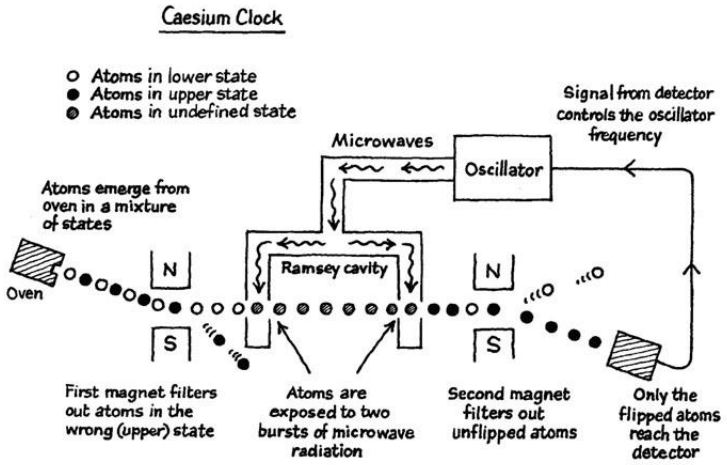


Fig. Caesium clock

Cosmic radiation has been previously discovered in 1912 by Austrian Victor **Hess**$(1883-1964)$. The Cosmic radiations are special radiations that are composed of a particles like protons, electrons... there are many elementary particles that have been discovered for the first time in the cosmic rays.

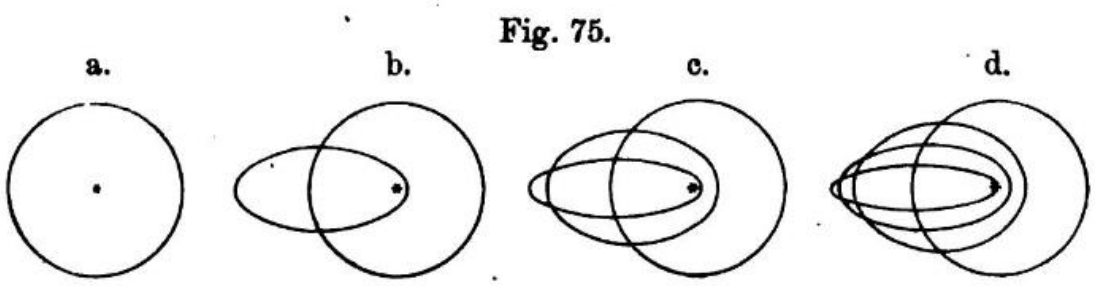
In 1911 the first Solvay conference took place in Brussels (Belgium), it was the idea of Belgian chemist Ernest **Solvay**$(1838-1922)$ to gather European scientists of the time to discuss the new science and its open problems.

Fig. the first Solvay conference (1911).

The orbitals

As we have said, the work of **Bohr** was guided by his effort to explain the laws of spectroscopy especially in the work of by **Balmer** and **Rydeberg**, i.e. to explain the spectrum of atoms. To each substance corresponds a spectrum, so **Bohr** succeeded to make a relation between his theory and the theory of spectra. The spectrum lines correspond to the atom energy levels. Later on, other physicists applied the idea to more complicated spectra.

Among the physicists that played a key role in quantum theory was German Arnold **Sommerfeld**$(1868-1951)$, he introduced new ideas, he for example introduced the *azitumal* number that we will see. He suggested that the orbits were not only circular but can be *elliptic*.



Nachdem wir jetzt die verschiedenen Möglichkeiten für die einzelne Bahn bei gegebenem $n + n'$ untersucht haben, können

Fig. elliptic orbitals.

The electron does not turn around the atom in a simple way, its trajectory is very complicated. The trajectory is composed of layers which are : shell, subshells and orbital. So we will identify the different layers around the nucleus as follows:

–*Shell:* it is the main trajectory of the electron around the nucleus. There can be many shells around the nucleus and they are given the alphabetic K, L, M, …they represent the energy levels. To every shell corresponds a principal quantum number

$$K \equiv n_1$$
$$L \equiv n_2$$
$$M \equiv n_3$$
$$\dots\dots.$$

as we will see. In every shell there are a maximum $2n^2$ of electrons.

–*Subshell:* every shell can be composed of one subshell or many subshells. We can cite 4 of them :

s(sharp), p(principal), d(diffuse), f(fundamental),

and there are other subshells like g(after f).

Shell	Subshell
K	s
L	s, p
M	s, p, d
N	s, p, d, f

– *Orbit:* every subshell is composed of trajectories called orbit or orbital.

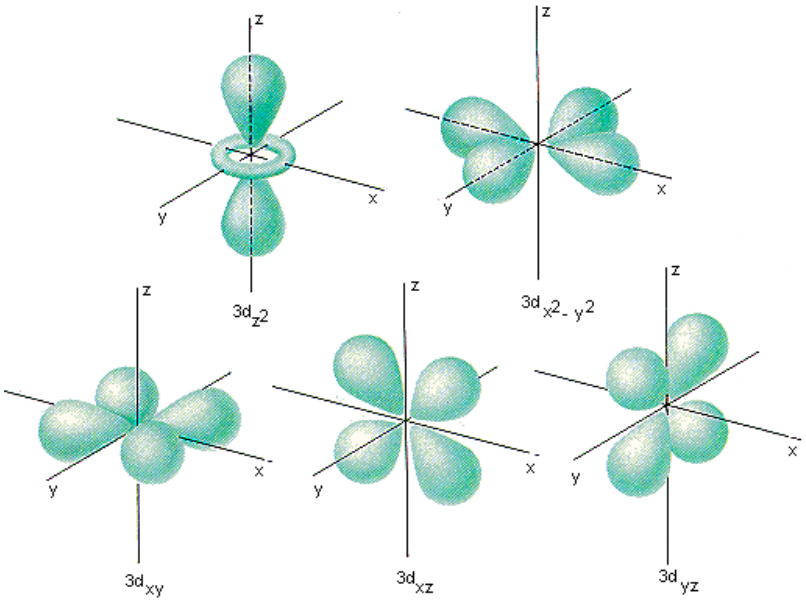


Fig. Every shell has a certain number of subshells.

So we have

$$orbital \subset subshell \subset shell$$

The number of shells gives special characteristics to each atom. Classical mechanics cannot explain how the electrons are turning around the nucleus. With a big speed, what if the electrons collide when turning in one orbital around the nucleus? Another problem was how the electrons are distributed on orbits around the nucleus. Now we move to the quantum numbers.

The quantum numbers

An electron has many parameters that we try to measure as : speed, angular momentum, spin, charge, energy. Except speed all the parameters are quantified. It is means that all the numbers are quantified. They have been given names

$$n, s, r, l$$

So researches came to the conclusion that a particle has four numbers (n, ℓ, m, s)

– *The principal quantum number* n : can take the positive integer values : 1,2,3, … it represents the energy levels Also in the orbits the state of electrons goes from low energy to high energy

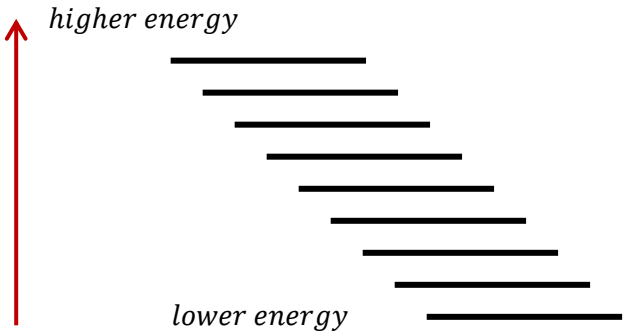


Fig. The energy distribution.

– *The angular momentum* ℓ: or azimuthal, introduced by **Sommerfeld** in 1915. it is the of the orbital spherical or elliptic it can take the positive integer values : 1,2,3, … . it always verifies the

$$0 \leq \ell \leq n - 1$$

where n is the principle quantum number. ℓ describes the shape of the orbital.

– *The magnetic quantum number m* : noted also m_l and has a relation with ℓ through the

$$-\ell \leq m_l \leq \ell$$

It the orientation of the orbitals in space and thus distinguishes them.

– *The spin quantum number s* : noted also s_ℓ , we have

$$s_\ell = -\frac{1}{2} \qquad \text{or} \qquad s_\ell = \frac{1}{2}$$

It describes the intrinsic angular momentum. It was introduced as we will see in 1924. German Alfred **Landé**$(1888 - 1976)$ found in 1925 a rule between the spin and momentum.

Since **Bohr** theory, the work of physicists focused is to implement these ideas on atoms. The atom they knew was hydrogen H with one electron and one proton. The atom existence was confirmed, researchers began the determination of atomic numbers. British Henry[47] **Mosley**$(1887 - 1915)$ used X-rays to determine the atomic number of many atoms. Other elements have been studied like alkali:

$$Na(11 \text{ electrons}), \quad K(19 \text{ electrons}),\ldots$$

so for an element is written as follows

[47] He died prematurely in action at the age of 27.

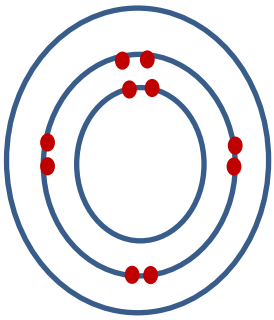


Fig. The first shell has always tow electrons.

The case $n = 1$ is called the *ground state*.

The electron configuration and the exclusion principle

So we will now apply all the precedent idea on atoms. Physicists worked between 1919 and 1926 to find how to fill the electrons in every atomic orbital and subshells, there are

Subshell	Number of electrons
n_1	2
n_2	8
n_3	16
n_4	32

Fig. Electrons in subshells according to the law $2n^2$

So the following description

$$1s^2 2s^2 2p^6 3s^2 3p3 \ldots$$

Means for $n = 1$ the first shell we have subshell s that has only and 2 electrons in it . For $n = 2$ we have 8 electrons, 2 in s only and 6 electrons in p. And every with the maximum number of electrons

$$1s^2 \rightarrow 2s^2 2p^6 \rightarrow 3s^2 3p3 \rightarrow \cdots$$

And here are configurations of the first 12 atoms in the periodic table:

H : $1s^1$
He : $1s^2$
Li : $1s^2 2s^1$
Be : $1s^2 2s^2$
B : $1s^2 2s^2 2p^1$
C : $1s^2 2s^2 2p^2$
N : $1s^2 2s^2 2p^3$
O : $1s^2 2s^2 2p^4$
F : $1s^2 2s^2 2p^5$
Ne : $1s^2 2s^2 2p^6$
Na : $1s^2 2s^2 2p^6 3s^1$
Mg : $1s^2 2s^2 2p^6 3s^2$

The preceding is made according to 2 rules:

– *Aufbau principle* due to **Bohr** who found it around 1920 (Aufbau a German word means build). To build the atom, the rule says that electrons are filled in order, so the lower level filled first then the outer levels.

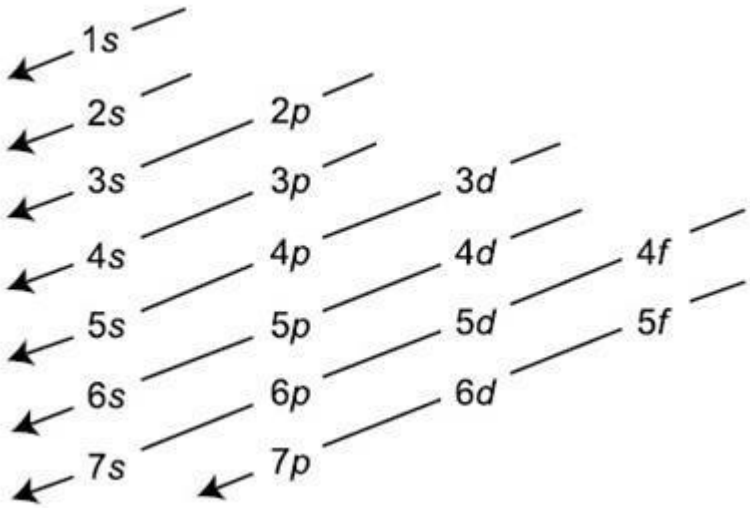


Fig. The Aufbau principle

–**Hund** rule : due to German Friedrich **Hund**$(1896-1997)$ it says that low *orbitals* energy are filed first then the higher ones. So orbitals have the same energy if they are of the principal shell. It used in chemistry.

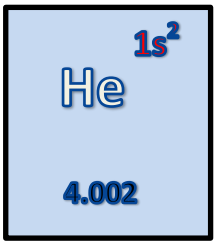


Fig. The element He with his configuration

Some of the consequences of the new configuration can be intriguing. One of the father of quantum mechanics, Austrian Wolfgang[48] **Pauli**$(1900-1958)$ came with the idea in 1925, that two electrons cannot have all quantic numbers equal. If so, they will exclude each other. This is the *exclusion principle*. If

$$(n,l,m,s)=(n',l',m',s')$$

we have

$$n=n',\qquad l=l',\qquad m=m',\qquad s=s'$$

According to this principle, two electrons cannot occupy the same quantum state. So only a finite number of specified electrons can be in every shell. We will see that this law works for *fermions*(electrons,…) only and not for *bosons*(photon,…).

[48] To **Pauli** is attributed the known expression "not even wrong" used when one's of a ridiculous statement.

The correspondence principle

In 1918 , **Bohr** published a paper under the title : *"On the quantum theory of line spectra"* . In his paper he remarks that the results predicted by classical electrodynamics must be still valid at the small frequencies ν. In so when

$$n \longrightarrow +\infty$$

the quantic mechanics tend to classical physics. This the same idea of Planck before him, namely that quantum to classical mechanics if

$$h \longrightarrow 0$$

the energy are not all quantized and we find ourselves in the classical case. This principle played an important role from 1919 until to breakthrough of 1925. The correspondence principle has been a systematic guessing guide used by physicists.

Fig. the second Solvay conference (1913).

Exercises

1)if an atom has the energy fo

1)knowin that $lim_{x\to 0}\frac{\sin(x)}{x}=1$ find:

i) $lim_{x\to 0}\frac{1-\cos(x)}{x}$ ii) $lim_{x\to 0}\frac{\sin(ax)}{bx}$ iii) $lim_{x\to 0}\frac{\tan(x)}{x}$

1) find $lim_{x\to+\infty}\, xsin\left(\frac{1}{x}\right)$.

1) find : $lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$

1) find $lim_{n\to\infty}\, cos\left(\frac{1}{n}\right)^n$.

1) find $lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$.

1) find : $lim_{x\to a}\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a}$.

1) find $lim_{x\to 0}\frac{2^x-1}{x}$.

1) calculate the rest energy E_0 of an electron in Joules.

1) calculate the rest energy E_0 of an electron in electronvolts.

1) find the electron speed expression around the nucleus.

1) find the radius and energy of the following atoms

i) $Al:\ Z=26$

ii) $Si:\ Z=28$

iii) $Pi:\ Z=30$

1) find the wavelength transition from $n=2$ to $n=4$.

1) can the following quantum numbers be true?

i) $n=1,\quad l=1,\quad m=2,\quad s=-\frac{1}{2}$

ii) $n=2,\quad l=-1,\quad m=-2,\quad s=\frac{1}{2}$

iii) $n=3,\quad l=2,\quad m=1,\quad s=-\frac{1}{2}$

1)find the configuration of the following atoms:

i) $S\ :Z=32$
ii) $Cl:Z=34$
iii) $Ar\ :Z=36$
iv) $K\ :Z=38$

1) find out the expression of E_n given in the text(use $K + V$).

ABSTRACT TOOLS

The physicist who states a law of nature with the aid of a mathematical formula is abstracting a real feature of a real material world, even if he has to speak of numbers, vectors, tensors, state-functions, or whatever to make the abstraction.

Hilary **Putnam**

As geometry is tightly linked to trajectory of particles in mechanics, today in physics, to study these trajectories, physicists need many sophisticated objects that come from differential geometry like tensors. Tensors as vectors and matrices, are very important in geometry and modern physics. Tensor analysis has becomes an essential tool in many situations like relativity, Yang-Milles theory, string theory,… Tensors exited already in the geometry work done by German Frederick **Gauss**, after many other works like those of Swiss Marcel **Grossmann**$(1878-1936)$, the word *tensor* was first introduced by German Woldemar **Voigt**$(1850-1919)$ in 1898.
The trajectory of a particle is considered as a *manifold*(curves and surfaces are examples of manifold). So our target here is to understand the geometry of the space. Also we can apply the method of calculus to manifolds by integrating, derivating,…new objects called *differential forms*.

Tensor theory

Tensors belong to the realm of linear algebra or exactly to multi-linear algebra. A tensor Γ roughly generalizes a vector, a vector represents a force, but in many situations a vector does not suffice to represent a given physical concept. So we need something stronger. In a general language a tensor Γ generalizes some that stuff we work all the time : numbers,

vectors, matrices. So a vector is a special case of a tensor; more precisely:

−a number Γ is a tensor of *rank 0.*
−a vector Γ_i is a tensor of *rank 1*.
−a matrix Γ_{ij} is a tensor of *rank 2*.

What is new here is the tensor of rank 3 noted Γ_{ijk} which is a proper tensor. The word "tensor" comes from latin verb "tendere" that means to stretch. It measure the invariance in terms of coordinates. So the components of a tensor can be *covariant* or *contravariant* . A tensor of this rank can be covariant, noted

$$\Gamma_{ijk}$$

or contravariant noted

$$\Gamma^{ijk}$$

and the relation in terms of a matrix a_{ij} is

$$\Gamma^{ijk} = a_{ij}\Gamma_{ijk}$$

Generally, a *rank−n tensor* of dimension m has m^n components. Unlike a vector, a tensor can have many directions. Tensors help to present many things in geometry and physics, for example :

− *Stress*: it is the pressure $\sigma = \frac{F}{S}$ on a surface unit. Stress cannot be presented by only a vector (this precisely for what **Voigt** put a tensor for the first time) .

− *Strain:* is the deformation caused by a force applied on a body.

Unlike a vector, a tensor has many directions. A tensor is represented by the symbol Γ^i where the number of letter i is the rank. As an for example

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

The σ_{ii} are the components of a force, so the tensor σ presents 12 forces. There is another concept called ***multivector***, every multivector is a tensor.

In formal language, a tensor is defined as follows

Let V a linear space, a k-tensor is a multilinear function from $V_1 \times V_2 \times \ldots \times V_k$ to the set of real numbers $\mathbb{R}$

Like vectors and matrices, we can also perform on tensors some operations. We define addition

$$A^{ij} \pm B^{ij} = C^{ij}$$
$$A_{ij} \pm B_{ij} = C_{ij}$$

we make a difference between vectors and components by writing

$$x^i \quad \text{and} \quad x_i$$

We use x_i for vectors and x^i for the vector components. As the same with the direct sum between two vectors v_1 and v_2 , we can form a new tensor called the ***tensor product***

$$v_1 \otimes v_2$$

and between two vector spaces we can perform a new operation called the ***tensor product*** as follows

$$V_1 \otimes V_2$$

to get a bigger new vector space.

There is a simple trick to sum called **Einstein** convention. Instead of writing summation $\sum_{i=1}^{3} a^i c_i$ we write simply the following

$$x = a^i c_i$$

which means

$$a^1 c_1 + a^2 c_2 + a^3 c_3$$

and we use often the **Kronecker** symbol

$$\delta_{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$$

We have then $I = [\delta_{ij}]$. With Kronecker symbol we define a new object, it is called Levi-Civita symbol ε_{ijk}.

Manifolds

Differential geometry is the part of geometry began mainly by **Euler** that studies curves and surfaces by the language of infinitesimal calculus. We study usually curves and surfaces with the following things:

–if a curve is regular, simple.
–tangent to a curve.
–length of a curve. The length of a curve between two points(a, b) is given by :

$$\int_a^b \sqrt{1 + f'^2(x)} dx$$

–intersection of two curves.
–intersection of a curve and a surface, intersection of two surfaces.
–how to calculate the distance between two points on a surface.
–curvature of a curve and of a surface.

Afterwards **Gauss** comes with his amazing *Theorema Egregium*. Modern differential geometry used by physicists speaks in terms of manifolds. The first person to introduce them was German Bernhard **Riemann**$(1826 - 1866)$. A manifold M is the abstract generalization of a curve and of a surface. To define a manifold M, we try to find a map

between M to $\mathbb{R}^n$, which means that although M can be very complicated, it is locally like $\mathbb{R}^n$. A manifold M is a hausdrff topological space where very $p \in M$ we can a map to .

A plane in $\mathbb{R}^3$ is also a manifold. we can make map between two manifold M and N

Physicists consider the trajectory of a particle as embedded in a *manifold*. There are general manifold called topological manifolds and other more sophisticated one called differential manifolds. Then the main study of very special ones called smooth manifolds. It uses differential apparatus mainly originated in the work of **Riemann**. The concepts are :

−metric , it is the distance between two points. We can define many different metrics: Riemannian metric, Lorentz metric. The Riemann metric can be a dot product defined on the tangent space. A Riemannian manifold is noted (M, g) where g is the metric.

− the particles moving on M. The tangent space T_pM to a manifold M at the point p is considered by physicists as the space of velocities of particles moving on M.

−parallel transport and holonomy on a connection , which is related to the curvature.

−connection on vector bundles. The **Levi-Civita** connection is the unique affine that Riemann tensor. The Curvature of a connection is

$$dA + A \wedge A$$

where d is the exterior derivative and $A \wedge A$ is the *exterior*[49] *product*(or wedge product). When $n = 2$ it is defined

$$a \wedge b = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

for

$$(a_1 i + b_1 j) \wedge (a_2 i + b_2 j) = a_1 a_2 i \wedge i + a_1 b_2 i \wedge j + b_1 a_2 j \wedge i + b_1 b_2 j \wedge j$$

$$= a_1 b_2 - a_2 b_1$$

[49] A part of *exterior algebra* created by German Hermann **Grassmann**$(1809 - 1877)$ an developed further by French Elie **Cartan**$(1869 - 1951)$.

The wedge product of two vectors is the area of the surface between them(the surface is oriented and therefore area is signed).

To every gradient (vector field) we can associate a differential form. It is any of

$$Adx + Bdy + Cdz + \cdots$$

a 0-form is just a function of n variables $f(x_1, x_2 \ldots, x_n)$. The product of two differential forms

$$(Adx + Bdy + Cdz) \wedge (Ddx + Edy + Fdz)$$

is just the vector(cross) product of two vectors.
Unlike vector product, $\wedge$ is associative . We can prove that $a \wedge a = 0$ and $a \wedge b = -b \wedge a$.
When the curvature is infinite there is a ***singularity***.

–fiber bundle. A vector bundle is a special case of a fiber bundle.

Metric

As we have stressed, the concept of manifold is due to **Riemann** and the idea behind it is the generalization of curve and surface to space dimension $n \geq 4$, where we cannot draw but we can understand the new abstract structures and manipulate them with the aid of equations. The idea is if we know how to calculate the distance (length) between two points we can understand the nature of geometry we are dealing with.

We mean with a metric a distance, that permits on a manifold to:

– measure the distance between points on a manifold,.

– measure the area of a surface.

– measure the geodesics.

– measure the curvature.

– measure the angles.

– measure the volume.

– measure the length.

There is a metric called ***Riemann metric*** founded on inner product. There are many metrics defined on a manifold: Quadratic metric. We know that the distance between two points is given by :

$$ds^2 = dx^2 + dy^2 + dz^2$$

Riemann generalized this metric, his metric is defined via an inner product

$$g \ : \ T_pM \times T_pM \ \longrightarrow \ \mathbb{R}$$

by

$$ds^2 = g_{ij}dx^i dx^j$$

where $g_{ij}dx^i dx^j$ is seen as a sum . We see that he generalized the curvature from Euclidian metric to new metric and the curvature parameters will be presented by tensor like Γ_{ij}. For example, the curvature by a tensor on the tangent space is given by

$$\frac{\partial \Gamma_j^i}{\partial \Gamma_j^i} - \frac{\partial \Gamma_j^i}{\partial \Gamma_j^i} + \Gamma_j^i \Gamma_j^i + \Gamma_j^i \Gamma_j^i$$

and in terms of vector fields X and Y

$$\nabla_X \nabla_Y - \nabla_X \nabla_Y + \nabla_{[xy]}$$

it measures the non-commutatively of the covariant derivative. The should verify the **Bianchi** identity(the paper of **Bianchi** was related to Riemann curvature)

$$R^{\alpha}_{\beta\delta\gamma} + R^{\alpha}_{\gamma\delta\beta} + R^{\alpha}_{\delta\alpha\beta} = 0$$

Curvature

Modern physics, as general gravity, use the language of differential geometry, we will see also in the gauge theory which is implied in the dynamics particles as well.

If the curve is given by a parametric representation $\gamma(t)$ in a form of a vector the length is given by : $\int_a^b |\gamma'(t)| dx$. A *curvature* of a curve noted by the symbol κ in geometry, measures how much a curve or a surface is bend. A line in the plane $\mathbb{R}^2$ and a plane in the space $\mathbb{R}^3$ have both curvature $\kappa = 0$. If a curve C is given by a functional relation $y = f(x)$, then the curvature of C at the point (x_0, y_0) is given by the formula

$$\kappa = \frac{f''(x_0)}{[1 + f'(x_0)]^{3/2}}$$

the curvature of a circle is constant.
an important result in theory of curves is that a *regular* curve is *uniquely determined* by its curvature κ and *torsion* τ, through the known Frenet-Serret equations

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \end{cases}$$

For a surface the situation is different, a surface has an infinity of tangents at a point. To uniquely determine a surface we use *forms*,

$$\begin{cases} Edu^2 + Fdudv + Gdv^2 \\ Ldu^2 + 2Mdudv + Ndv^2 \end{cases}$$

where E, F, G, L, M, N are functions linked to u and v.

In 1827 , **Gauss** proved his Theorema Egregium (remarkable theorem) consisting in the fact that the curvature of a surface in is intrinsic and independent of the coordinates. This idea was generalized to manifolds by his student German Bernhard **Riemann. Riemann** did gigantesque work in geometry by founding *Riemannian geometry* a collection of very

original ideas. This geometry is the foundation of general relativity.

Fiber bundle

Modern differential geometry is based on the *fiber bundle* theory. We began by saying that a tangent line to a curve and a tangent plane to a surface at a point is generalized to *tangent space* at a point. The tangent space T_pM to a manifold M at the point p is viewed as the space of velocities of particles moving on M. The set of all tangent spaces is the *tangent bundle* TM defined by

$$TM = \bigcup T_pM$$

Exercises

1) Show that: $\alpha = arctan\left(\frac{y}{x}\right)$.

1) how many components has a tensor of rank :$n = 0,1,2,3$ in the standard dimension 3.
1) let $v_j = \sum_{i=1}^{3} a_{ij}\, u_i$ and $u_j = \sum_{i=1}^{3} b_{ij}\, v_i$. Show that : $\sum_{j=1}^{3} a_{ik}\, b_{kj} = \delta_{ij}$.
1) find :
i) $3\delta_{21} + 3\delta_{11} - \delta_{13} =?$
ii) $-\delta_{31} + \delta_{31} + \delta_{22} =?$
iii) $\delta_{22} + 2\delta_{12} - 5\delta_{11} =?$

1) show that : $\sum_{j=1}^{3} \delta_{ij}\, a_j = a_i$.
1) show that $a \wedge a = 0$ by two methods.
1) find : $(Adx + Bdy + Cdz) \wedge (Ddx + Edy + Fdz)$.
1) Show that $dv_1 \wedge dv_2 \wedge dv_1 = 0$.
1) show that the equation $x + y + z + 1 = 0$ cannot be an equation of a line in $\mathbb{R}^3$.
1) Let the points $a(1,0,-1), b(0,0,1)$ and $c(-1,-1,0)$ belong to a plan. Find its equation(use both vector and dot products).
1) Let the surface $f(x,y,z) = c$ the equation of a surface show that ∇f is perpendicular to it.
1) Show that $x(t) = (1+t)e_1 + (t^2+3)e_2$ is a *regular* representation of a curve, and is a parabola.
1) Show that $x(t) = te_1 + (1+t^2)e_2 + (t-1)^3 e_3$ is a regular parametric representation of all t and find the representation onto the x_1x_2 and x_1x_3 planes. These are twisted (or space) curves, in French: *courbe gauche*.
1) Calculate the curvature of the curve $f(x) = ax^2 + b$ at $(0,0,)$.
1) Calculate the curvature of the curve $f(x) = ax + b$ at every (x_0, y_0).

1) Let $f(x) = x + 3$. Calculate the length s of the curve from $x = 1$ to $x = 2$.
1) Find it by Pythagoras theorem.

1) Let $f(x) = x + 1$. Calculate the curvature of the curve at $x = 2$.

THE THEORY OF GRAVITY

…General relativity, however, was by no means the work of "a lone genius". First of all, one of the central claims we will argue for this volume is that Einstein in developing his theory for the gravitational field was guided predominantly not by considerations of mathematical elegance, as he would maintain in his later years, but by and elaborate analogy with Maxwell-Lorentz theory for the electromagnetic field reformulated I the four-dimensional formalism of special relativity by Hermann Minkowski, Arnold Sommerfeld, Max Laue and others. Moreover, although he did not always acknowledge this in print, Einstein, as he was developing his theory, received substantial help and feedback from a number of individuals, some well-known, others long forgotten, such as Paul Bernays, Michele Besso, Paul Ehrenfest, Adriian D. Fokker, Erwin Freundlich, Marcel Grossmann, Paul Herz, Friedrich Kottler, Tullio Levi-Civitta, Hendrik A. Lorentz, Gunnar Nordstrom and Moritz Schlick.

Michel ***Janssen***

In 1915, Albert **Einstein** announced his *general theory of relativity*, the purpose of the theory was to explain *gravity*. **Newton** has found that the force of gravity between two bodies (m_1, m_2) can be evaluated by the formula $F = G\frac{m_1 m_2}{R^2}$. The language in which the new theory has been written is the language of tensors. After **Gauss** and his followers, the tensors have been used further in the work of Italian Gregorio **Ricci-Curbastro**$(1853 - 1925)$, he has even given an adequate definition. Around 1900, another Italian Tullio **Levi-Civitta**$(1873 - 1941)$ published his book: " *Absolute tensor calculus*" in French language. **Levi**-**Civitta** taught **Einstein** the new calculus through correspondence. Nowadays tensor analysis is a part of multilinear algebra.

The theory of relativity depends also on the *equivalence principle*.

The equivalence principle means the equivalence of inertial mas and gravitational mass

We have seen that

$$m_i = m_g$$

We have seen that mass can be inertial presented by Newton law $F = ma$, or gravitational in the relation $F = G\frac{m_1 m_2}{r^2}$. In this latter case it is like charge in Coulomb law.

Evaluating gravity

Newton showed that gravity is responsible of both falling of an apple to the ground and of the movement of planets around the Sun. Gravity makes our life possible on Earth. We know that earth attract us to the ground and also and attract moon. Why does moon fall on earth ? because the force is not strong enough. But how is it possible that this big numbers of planets and stars are swimming in space through this law ?

Before the end of XIX century, scientists were occupied by the problem of calculate the gravity force of traction of a body. French mathematician Pierre-Simon **Laplace**$(1749 - 1827)$ working to *celestial mechanics*, that helped in discovering new planet turning around the sun. **Laplace** considered that gravity of a potential gravitational verifies to the equation

$$\Delta u = 0$$

called the Laplace equation, it is the divergence of the gradient of u. Every *analytic* complex functions are harmonic. The solutions of this equation are harmonic functions. The **Laplace** equation is a special case of another equation, it is the **Poisson** equation found in 1813

$$\Delta u = -\frac{\rho}{\varepsilon_0}$$

when $\rho = 0$ (vacuum). The **Poisson** equation is equivalent to of **Gauss** law $\nabla \circ E = \frac{\rho}{\varepsilon_0}$ in electrostatics, for in field $E = -\nabla u$ we have

$$\nabla \circ E = \nabla \circ (-\nabla u) = -\Delta u = -\left(-\frac{\rho}{\varepsilon_0}\right) = \frac{\rho}{\varepsilon_0}.$$

For gravitational gravity Poisson also has

$$\Delta u = 4\pi G\rho$$

The search for gravity force leads to the famous *Dirichlet problem*[50] : find a function u in a domain D such that

$$\begin{cases} \Delta u = 0 & inside\ D \\ u = f & on\ the\ boundary\ of\ D \end{cases}$$

The curved space-time

What is exactly the nature of gravity ? in 1907 **Einstein** through a thought experiment, found that during a free fall weight can disappear, so that gravity can be zero:

I was sitting in a chair in the patent office at Bern when all of a sudden a thought occurred to me :"if a person falls freely he will not feel his own weight" .I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation.

An experiment in which this prove propriety can be proved. On 25th of November of 1915, Albert **Einstein** submitted his paper "*The field equations of gravity*" in Berlin. So the General theory of relativity on :

- principle of equivalence.
- space time curvature
- rejection of absolute space.

[50] See the appendix of Umberto **Bottazzini** book : *The Higher Calculus*.

The rejection of absolute space was not new, what was new in this **Einstein** is the nature of gravity. He introduced a new idea about the concept of *space-time* introduced by **Poincaré** before that in the theory of relativity.

His view is that gravity which attracts planets and feel the space is very abstract and it is in reality the curvature of space-time. Time is linked to gravity; in fact time depends on gravity. On earth when time passes more slowly in low places than in high places. The idea is that

Gravity makes time slow

Einstein used **Minkowski** space an aided by his Marcel **Grosamn**$(1878-1936)$. So in tensors language, **Einstein** equation called *Einstein field equations* can be written

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu}$$

$\kappa=\frac{8\pi G}{c^4}$ is the gravitational constant . The three terms $g_{\mu\nu}$, $G_{\mu\nu}$ and $T_{\mu\nu}$ are tensors. $g_{\mu\nu}$ is the **Ricci** tensor. The tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}\mathrm{R}g_{\mu\nu}$$

$R_{\mu\nu}$ is the **Ricci** curvature tensor. $T_{\mu\nu}$ is the stress-energy tensor. R is the scalar curvature. Λ is the cosmological constant. When $\Lambda=0$ we have

$$R_{\mu\nu}-\frac{1}{2}\mathrm{R}g_{\mu\nu}=\kappa T_{\mu\nu}$$

The curvature of space-time is

$$R_{\mu\nu}-\frac{1}{2}\mathrm{R}g_{\mu\nu}$$

means

$$Curvature = stress\ energy$$

Field equations are like Newton second law that establishes a relation between space-time, energy. The equations are not linear. **Bianchi** identities due to Luigi **Bianchi**$(1856 - 1928)$ discovered in 1902

$$R^{\alpha}_{\beta\delta\gamma} + R^{\alpha}_{\gamma\delta\beta} + R^{\alpha}_{\delta\alpha\beta} = 0$$

where R is the **Riemann tensor**. **Riemann** work of the year 1854 on geometry made generalization. Later it has found its use in the theory of general relativity. The space-time is just an abstract manifold of dimension 4. Its metric $g_{ab}$is Riemannian , it enables us to measure the curvature of the space time. The matter is presented by tensor fields. **Hilbert** also was led to the same equations, so they are often called also **Hilbert-Einstein** equations. He has identified the Riemann scalar as the Lagrangian for the metric field before **Einstein** had hit upon the field equation that follow immediately from this Lagrangian.

Solutions

in 1911 **Einstein** conjectured that light itself would be deviated by a planet gravity according to the curvature of space-time which presented by the term

$$R_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu}$$

The significance of this is that the rays can be deflected by the gravitational force so the bending of light by the planet gravitational fields. In the year 1919 , during an eclipse the conjecture was confirmed. English Astronomer Arthur **Eddington**$(1882 - 1944)$ was leading an expedition to observe the possible starlight bend on May 29^{th} 1919.

So after the equations have been given, are there any solutions? Like string theory, general relativity is difficult to prove experimentally and **Einstein** did not find a solution. Many scientists began to look for possible solution and among them:

–German Karl **Schwarzschild**$(1873 - 1916)$ who was the first, in 1915 , to find exact solutions for the field equations. He died in WW II. It is given in the form of a metric :

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}} + r(d\alpha^2 + sin^2(\alpha)d\theta^2)$$

–Hungarian Cornelius **Lanczos**$(1893 - 1974)$.

–Dutch Willem de **Sitter** $(1872 - 1934)$.

–Dutch Willem van **Stockum** $(1910 - 1944)$.

–New Zealand Roy **Kerr** $(1943-)$ who in 1963, found his metric and geometry as exact solutions in a form of black hole.

In 1922 , Russian Alexander **Friedmann**$(1888 - 1925)$ from the equations that govern the theory of expansion, It is also given in a metric form

$$c^2 d\tau^2 = c^2 dt^2 + R^2(t)(dr^2 + S_k^2(r)d\psi^2)$$

Mach principle

Physicists, through many experiments, have been led to discover many proprieties of motion like the inertia, centrifugal forces, dragging effects…. Newton's *bucket experiment* postulated that space is absolute. The problem with this result was this : why the water remains concave after the rotting stops?

Mach as we know criticized Newton's absolute space. Mach ideas were known at the beginning of the XIX century, and German Friedrich **Schlick**$(1882 - 1936)$ coined the expression " Mach postulate" in his book of 1915, it was **Einstein** who coined the expression *Mach principle.* According the principle, says *Britannica,*

The inertial forces experienced by a body in non-uniform motion are determined by the quantity and distribution of matter in the Universe.

we do not know if the principle is true, but at least a non-provable principle one. The is how can we the problem of reference of a motion? What relation has Mach's principle with general relativity ? **Einstein** has made this principle as the start to explore gravity in the theory of general relativity, but afterwards he abandoned it because of the consequences of his field equations.

Predictions

Einstein considered the equivalence principle also in general relativity. We do not know if the Universe is flat or curved(spherical, oval). The observable Universe has the diameter of 96 billion light years and contains 10^{11} galaxies. We cannot, using the methods of modern science, know if the Universe is finite or infinite.

So after a lot of trials and errors and contributions by physicists and astronomers, general theory of relativity predicted:

–Existence of gravitational waves. **Einstein** predicted gravitational waves in his theory and later physicist postulated an associated particle like photon called *graviton* which is still hypothetical. The gravitational waves were first proposed by **Heaviside** in 1893 and by **Poincaré** in 1905, they have been detected in 2015.

–Light is Doppler shifted in the gravitational field.

the latter prediction means that the falling light is Doppler-shifted in a gravitational field, and this was tested by the Pound-Rebka experiment in 1960 by Canadian Robert **Pound**$(1919-2010)$ and American Glen **Rebka**$(1931-2015$.

–The big bang[51] and the Universe expansion.

Physicists had found that they can adjust the equation modeling the general relativity such that one can infer that the Universe was expanding. The original field equation emitted by **Einstein** was of the form

[51] The expression "Big bang" was coined by English astronomer Fred **Hoyle** in 1949 as a pejorative term against the idea that the universe had a beginning.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

The constant Λ is called the *cosmological constant*, it was added by him in 1917 to ensure that the Universe be *static*. Later he removed it for evidences, so when we put $\Lambda = 0$ the situation changes dramatically. This because, as discovered by many astronomers, as American **Hubble**, that the Universe could not be static. Another scientist was Belgian Georges **Lemaitre**$(1894 - 1966)$ concluded that the Einstein equations imply that the Universe is expending, another proof of the velocity in which galaxies recede from our galaxy. It was in 1998 that an international team of cosmologists$(High - Z\ Supernova)$ working on supernovae discovered that $\Lambda > 0$ which means that the Universe expansion is not uniform but accelerating.

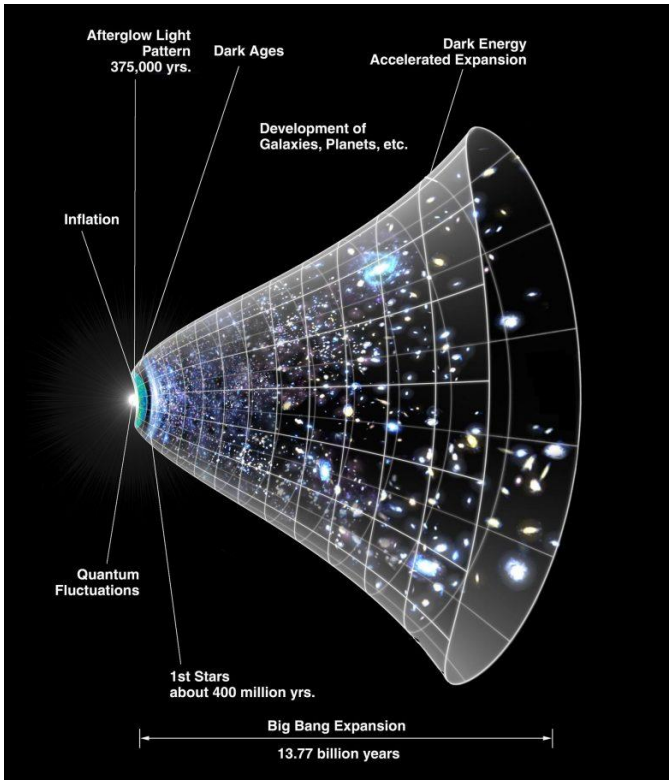


Fig. Beginning and expansion of the Universe

–Black holes.

The theory of general relativity predicts the existence of some strange phenomena. A *black hole* is a place in the Universe where the force of gravity is very strong. Places in outer space with strong gravity in such a way that light cannot escape was considered first by

English John **Michell**$(1724-1793)$ and French Pierre **Laplace**$(1749-1827)$. In terms of the new theory, it is seen as the singularity solution of the field equations.

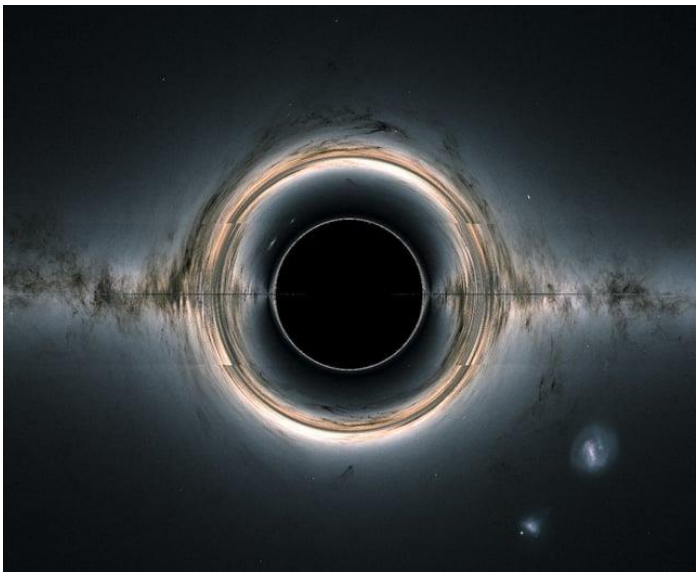


Fig. A sketch of a black hole by French astrophysicist **Luminet**.

Black holes are extensively studied object by astrophysicists and are still very mysterious places. It is believed now that every galaxy in our Universe has one at its center. It is also believed that black holes may have an essential role in the construction of galaxies and in the evolution of the Universe.

Another solution was given by Stephen **Hawking**$(1942-2018)$ in 1975 and Roger **Penrose**$(1931-)$ in a work with relation to *black holes*. In fact the **Schwarzschild** solution

$$ds^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2dt^2 + \frac{1}{1-\frac{2GM}{c^2r}} + r(d\alpha^2 + sin^2(\alpha)d\theta^2)$$

implies that the space-time curvature may become infinite, which gives a sort of *singularity*. When the **Einstein** equations have a singularity that means that the curvature is infinite, and in this case we say that the equations *break down*.

In 1998 Americans Saul **Perlmutter**$(1959-)$, Brian **Schmidt**$(1967-)$ and Adam **Riess**$(1969-)$ with important data given by the Hubble Telescope discovered that the universe expansion is accelerating. They even tried to calculate the Hubble constant(which

is different from the cosmologic constant)

$$H_0 = \frac{v}{D} = \frac{speed}{distance}$$

Because the universe expansion is supposed to slow down or stop because of the force of gravity, it has been proposed that something may be the cause of the expansion which we do not know. Astrophysicists believe that this accelerating expansion is fueled by a mysterious energy that they call *dark energy*.

Fig. An astronaut correcting Hubble mirrors in 1993.

Another mystery is the *dark matter*, a term introduced by Suisse astronomer Fritz **Zwicky**$(1898 - 1974)$ in 1933, it is thought that it fills the space and does not interact with light. Many constructions have been built on earth to detect the particles of dark matter without success, these sort of particles are called *wimp* (*weakly interacting massive particles*).

Exercises

1) Show that: $\alpha = arctan\left(\frac{y}{x}\right)$.

1) Show that if the following function are harmonic $(\Delta u = 0)$:

i) $u(x, y, z) = x + y + z$

ii) $u(x, y, z) = xyz$

1) how many components has a tensor of rank :$n = 0,1,2,3$ in the standard dimension 3.

1) let $v_j = \sum_{i=1}^{3} a_{ij}\, u_i$ and $u_j = \sum_{i=1}^{3} b_{ij}\, v_i$. Show that : $\sum_{j=1}^{3} a_{ik}\, b_{kj} = \delta_{ij}$.

1) find :

i) $3\delta_{21} + 3\delta_{11} - \delta_{13} = ?$

ii) $-\delta_{31} + \delta_{31} + \delta_{22} = ?$

iii) $\delta_{22} + 2\delta_{12} - 5\delta_{11} = ?$

1) let δ_{ij} the **Kronecker** symbol, show that : $2\delta_{21} + 3\delta_{22} + 4\delta_{23} = 3$**.**

1) show that : $\sum_{j=1}^{3} \delta_{ij}\, a_j = a_i$.

1) show that the equation $x + y + z + 1 = 0$ cannot be an equation of a line in $\mathbb{R}^3$.

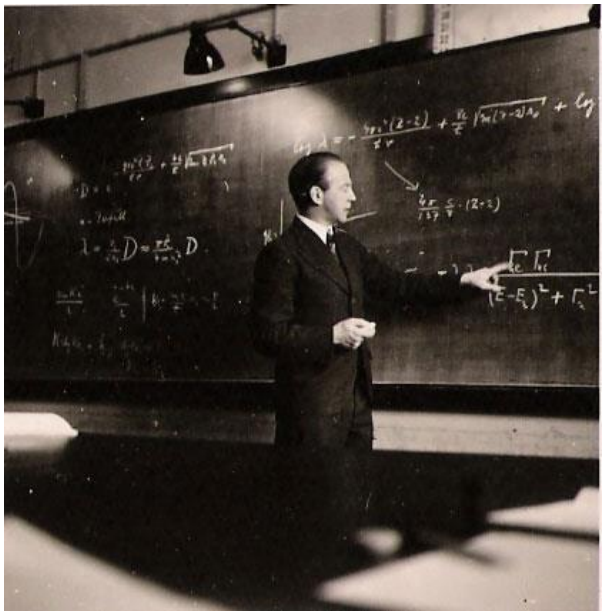
1) Let the points $a(1,0,-1), b(0,0,1)$ and $c(-1,-1,0)$ belong to a plan. Find its equation(use both vector and dot products).

1) Let the surface $f(x,y,z) = c$ the equation of a surface show that ∇f is perpendicular to it.

1) Calculate the curvature of the curve $f(x) = ax^2 + b$ at $(0,0,)$.

1) Calculate the curvature of the curve $f(x) = ax + b$ at every (x_0, y_0).

HEISENBERG MECHANICS



The first gulp from the glass of natural sciences will turn you into an atheist. But at the bottom of the glass God is waiting for you.

Werner **Heisenberg**

Historians of physics name *old quantum theory* the developments of quantum mechanics that took place from 1900 to 1925. In that last year, new principles were introduced, which are different from those of both classical mechanics and old quantum theory.

Fig. The third Solvay conference (1921).

The developments of the year 1925 are considered by physicists as " a turning point". Bohr theory was good but it needed more information to solve open problems. For example, Bohr theory predicted the states of the hydrogen atom, but could not predict the states of the next atoms like the helium atom He_4^2. New insight came from German Werner **Heisenberg** $(1901-1976)$who gave a new version of quantum mechanics called *matrix mechanics*. Thus he then German Ernest **Jordan** $(1902-1980)$ added a new theory to the old ideas by introducing *matrices* and *non-commutativity*.

Heisenberg mechanics

In the year 1925, **Heisenberg** laid the foundations of the new quantum mechanics. When he finished his PhD, he fall ill and moved to an island in the North Sea called Heligoland. There, he could finish his new ideas that would change the view of the newly created quantum theory. As the theory of **Bohr** succeeded for hydrogen atom but it did not work for heavier atoms, the problem at the time was the electrons transition from an orbit n to another orbit m he the formula. Heisenberg saw that old rules lacked physical foundation. The electrons are seen as harmonic oscillators, there anharmonic oscillator whose ODE is

$$x'' + \omega^2 x + \lambda x^3 = 0$$

and solution

$$x(t) = \sum_{n=0} a_{2n+1}\lambda^n \cos((2n+1)\omega t)$$

Heisenberg worked with $\omega\frac{h}{2\pi} = E_2 - E_1$ and Fourier series of a system

$$x(t) = \sum_{n=1} x_n e^{i(nt\omega)}$$

columns and rows contain the atom states, and $\omega\frac{h}{2\pi} = E_2 - E_1$ came to

$$(n,m) = \sum (n,k)(k,m)$$

(Heisenberg did not know that his is matrices product!!!) when in this new theory the product of two operators x and y that are associated to amplitude (later tmomentum and position)

$$x(t) = x_n e^{i(nt\omega)} \quad \text{and} \quad y(t) = y_n e^{i(nt\omega)}$$

as the product of two matrices M and P that are associated to momentum and position respectively,

$$[x(t), y(t)] = x(t)y(t) - y(t)x(t) = \alpha$$

Heisenberg was astonished to find that

$$x(t)y(t) - y(t)x(t) \neq 0$$

so they do not commute : $x(t)y(t) \neq y(t)x(t)$,

Heisenberg was not very happy with this. Heisenberg paper was also difficult to read. We quote here Steven **Weinberg** $(1933 - 2021)$

I have tried several times to read the paper that Heisenberg wrote on returning from Helgoland, and, although I think I understand quantum mechanics , I have never understood Heisenberg's motivations for the mathematical steps in his paper.

Max **Born** asked **Pauli** to help him to work out **Heisenberg** ideas but he refused. Afterward he and **Jordan** began to develop Heisenberg work, and they realized that the $x(t)$, $y(t)$ were matrices. **Heisenberg** did not know what a matrix was!! And he introduction of matrices

$$[x(t), y(t)] = x(t)y(t) - y(t)x(t) = i\frac{h}{2\pi}I$$

Or modern

$$[A,B] = AB - BA = i\frac{h}{2\pi}I$$

where I is the identity matrix. The multiplication is not commutative otherwise $i\frac{h}{2\pi}I = 0$ which is impossible.

The columns and rows of these matrices are composed of complex numbers. In this theory every propriety of a particle is presented by a matrix :

–the position $x = A$,
–the speed $\frac{dx}{dt} = B$,
–the momentum $p = C$,
–the energy $E = D$.

And a matrix is

$$\begin{bmatrix} (00) & (01)\cdots & (0m) \\ \vdots & \ddots & \vdots \\ (00) & \cdots & (0n) \end{bmatrix}$$

ω is the ***frequency*** and the ***amplitude***. So to the coordinate of an electron as in classical mechanics

$$\sum e^{ix}$$

And the aim is to called observables: energy, amplitude of transition and frequency. The is the law of multiplication

$$xy(n,m) = \sum x(n,m)y(n,m)$$

arrays in $xy(n,m) \neq yx(n,m)$. With

$$x(n,m)x(n,m) = |x(n,m)|^2$$

Is the probability that an electron goes from n to m. **Heisenberg** applied his "matrix mechanics" to the quantum states and to many particle atoms like helium. Later **Pauli** introduced his matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with relation to the concept of *spin*.

The spin

A *quantum system* is a name given to a particle or more, whose dynamics is studied by the physicist. A quantum system can be a nucleus, an electron… A *quantum state* is the vector that contains information on the particle. We quantum state for an electron for example, it is usually presented by the wave function. An electron has some parameters associated to it :

- Magnetic moment
- Lorentz force
- Angular momentum
- Intrinsic angular momentum

In English language "*to spin*" means "*to rotate*" around something, the spin of an electron is simply its intrinsic (independent of the rotation of the electron) angular momentum, we know that in classical mechanics the angular momentum is given by the formula $\mathcal{L} = r \times p$. So as a particle turning around the nucleus, it appears that it turns around itself, so the difference between angular momentum and intrinsic angular momentum.

After Goudsmit and Uhlenbeck had discovered the magnetic moment and the spin of the electron, almost all the difficulties of the earlier quantum theory disappeared, so that today we are in possession of a mechanical system which almost entirely satisfactory.

In 1925, German Ralph **Kronig**$(1904 - 1995)$ imagined the electron may rotate in space around itself, he told the idea to **Pauli** then with Samuel **Goudsmit** $(1902 - 1978)$ and

George **Uhlenbeck** $(1900-1988)$ introduced the spin of electron, it can be by

$$\vec{s}=\frac{h}{4\pi}\vec{\sigma}$$

where σ is a ***Pauli matrix***. When the electron turns around the atom it creates a magnetic field(as current in wires) so the spin is quantified. The concept of spin has solved a bunch of persisted problems, in the words of von **Neumann** in 1933

Then physicists imagined an electron should have either a *spin down*

Fig. The spin down of a particle

or a *spin up*

Fig. The spin up of a particle

We have learned that the orbits were given names:

$$n,\quad l,\quad m,\quad s$$

the fourth number s is the spin number. For the electron, the spin has two values

$$-\frac{1}{2}\quad \text{or}\quad \frac{1}{2}$$

The problem originated to explain spectrum of the alkali. We know that **Zeeman** effect is the mysterious split of the spectral lines and it turned out that it has a link with the spin in this way : it was the explanation of the effect in the new quantum theory that led to the concept of spin. It is the interaction between the magnetic field of the magnet and the angular momentum of the electron.

Spin	Particles
-1/2	Electron,
0	Higgs boson.
1	Photon, $W_{\pm}$, Z_0, gluon(?)
½	Electron, positron, proton, neutron, neutrino, quark, electron neutrino, muon, muon neutrino, tau, tau neutrino.
2	Graviton(?)

Fig. The spin of many elementary particles.

Spin is quantified like mass and charge. In 1932 , **Heisenberg** introduced the idea of *isospin symmetry* , to explain how the proton and neutron are bind in the nucleus.

The superposition principle

We have seen that **Bohr** correspondence principle says that for every quantum number the formulas of quantum theory must give results of classical mechanics and electrodynamics. Quantum mechanics also says that if we have two quantum states s_1, s_2 of a quantum system, the resulting of the two states is the sum

$$s_1 + s_2$$

So an electron, as a quantum system, can have many at the same time. We have already this situation in linear algebra, as many quantum laws are linear. for example in the domain of differential equations, if $y_1,\ y_2$ are both solution of $P(x, y, y') = 0$ then $y_1 + y_2$ also is a solution, more than that $ay_1 + by_2$ is also a solution, we have a *linear combination*. So we have superposed the two solutions to get one solution, it is the *superposition*

$$Superposition \ = \ linear \ combination \ of \ quantum \ states.$$

In the same way we can speak of superposition of two waves. So in quantum mechanics we have states. So an electron can have either a spin up or spin down, the superposition means that it has the two at the same time.

Almost all operators (like position,…)in quantum mechanics are ***unbounded***, which is a problem. We know for the theory of a finite dimension linear space, that to every finite matrix corresponds a linear map and to every linear map corresponds a finite matrix. In the case of an infinite dimension space, it is not the case. If the linear operator is unbounded in an infinite dimension, it may not represented by a matrix. In the case of unbounded operators, every self-adjoint operator is Hermitian but a Hermitian operator is not always a self-adjoint.

.

Exercises

1) show that the Pauli matrices are Hermitian.
1) let the Pauli matrices $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Calculate the following products:

$$\sigma_1\sigma_2, \qquad \sigma_2\sigma_1, \qquad \sigma_1\sigma_3, \qquad \sigma_3\sigma_1, \qquad \sigma_2\sigma_3, \qquad \sigma_3\sigma_2$$

1) show that the Pauli matrices are unitary.
1) let the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

calculate AB and BA. What do you remark?
1) let the matrices

$$A = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix}$$

why is $AB = BA$?
1) Let the spin $\vec{s} = \frac{h}{4\pi}\vec{\sigma}$. Find the eigenvalues of $\vec{s}$ for each σ.

1) Let y_1, y_2 solutions of $ay' = b$, show that $\alpha y_1 + \beta y_2$ is also a solution.

THE WAVE-PARTICLE DUALITY

The electron, as it leaves the atom,
crystalizes out of the Schrödinger mist
like a genie emerging from his bottle.

Arthur **Eddington**

Since light, which was considered a wave by physicists until the beginning of XX century, can behave as a particle, why then does not particle can behave as a wave? It is in these terms that French **De Broglie** thought in his doctorate theses in the 1920'. The photon (light) has no rest mass, it goes at the speed of light and has a momentum

$$p = \frac{E}{c}$$

Afterwards the experiment had showed that indeed it was the case, particles can be behave like waves. This marked the beginning of wave mechanics and the introduction of the celebrated **Schrödinger equation**. **Schrödinger** applied his wave function to the hydrogen atom H_2^1 with one electron and one proton.

Wave mechanics

Louis **De Broglie** $(1892-1987)$ was a French physicist who in 1924 came with a revolutionary new idea. After the development of photon theory, **De Broglie** expressed this idea in his doctorate dissertation under the title "*Research on the theory of quanta* " obtained at the French Academy of sciences in November, 25th 1924. Before **Heisenberg** theory in 1925, **De Broglie** has given his idea first in 1923 in 3 articles, then in his doctorate thesis , where he expressed his idea as follows

On peut concevoir que par suite d'une grande loi de la Nature, à chaque morceau d'énergie de masse propre m_0 soit lié un phénomène périodique de fréquence telle qu'on ait : $h\nu_0 = m_0c^2$.

Thus he associated to a particle a momentum p, and to every particle of matter a *real wave*.

From the known formula $p = \frac{h\nu}{c} = \frac{h}{\lambda}$, he postulated that the formula

$$\lambda = \frac{h}{p}$$

must hold. The former formula comes from the known formula $E = mc^2$ which is from special relativity:

$$E^2 = p^2c^2 \qquad \text{and} \qquad \left(\frac{hc}{\lambda}\right)^2 = p^2c^2$$

According to **De Broglie**, taking $E = mc^2 = h\nu$ we have

$$m = \frac{h}{c^2}\nu = constante \bullet \nu$$

So we see perfectly that to matter is associated a wave. So light has a momentum as expected, and with a wavelength

$$\lambda = \frac{h}{p} = \frac{h}{mv}\sqrt{1 - \frac{v^2}{c^2}}$$

where λ is the wave length of a particle called *the De Broglie wavelength*. The number $k = \frac{1}{\lambda}$ is called *wave number* a known parameter in classical wave theory. This gives $p = hk$. **De Broglie** also asserted following his theory that electrons can behave like the X-ray , so they could be diffracted by a crystal.

At first, the idea of **De Broglie** has been ridiculed by physicists as " French comedy" ,but in 1927 American Clinton **Davisson** $(1881 - 1958)$ at Bell Labs could observe a pattern of diffraction on a nickel crystal when bombarded by a beam of electrons. Another experience performed by English George **Thomson**$(1892 - 1975)$ at Aberdeen University, he obtained the same result by bombarding a fine crystal structure like gold, thus confirming **De Broglie** postulate. Many subsequent experiments showed the result by using even atoms and molecules :

- Pfleegor and Mandel(1967) with laser.
- Carnal and Mlynek(1991) with helium atoms.
- a team in Nebraska-Lincoln university(2012) with electrons.
- Fein(2019) with atoms.

But a question remained without an answer: if the particle can behave as light, which means as a wave, could the double-slit experiment for light waves performed by **Young** in 1801,be also performed using particles like electrons? In his course of physics which was written around 1965 $(p. 4 - 5)$, physicist Richard **Feynman** said :

…We should say right away that you should not try to set up this experiment (as you could have done with the two we have already described). This experiment has never been done in just this way. The trouble is that the apparatus would have to be made on an impossibly small scale to show the effects we are interested in. we are doing a "thought experiment," which we have chosen because it is easy to think about.

Before this and in 1961, German Claus **Jönsson**$(1930 - 2024)$ performed the experience by one electron at the university of Tubingen. It was only until in 1987 that a Japanese

team led by Akira **Tonomura** at Hitachi laboratory in Tokyo, an experiment that proved that electrons can show an interference wave pattern when ejected on a screen.

Another experiment, this time with atoms, was performed by O. **Carnal** and J. **Mlynek** in 1991. The experiment, as before, is a **Young**'s –type double-slit. They used helium atoms to pass through first one then two slits to arrive at a detector slit. In all those experiments, the idea is that particles travel as waves and arrive at the screen as particles.

Schrödinger equation

As physicists were sure that an electron is a wave, perhaps it is simpler to find the motion equation for a wave than for a particle!! So what remained was the equation that models the dynamics of the matter wave. In 1926, Austrian Erwin **Schrödinger**$(1887-1946)$ began developing a wave function to fit with the double wave−particle nature of a particle like the electron. The wave of **De Broglie** is associated to a *free* electron, not an electron linked to the atom. It is known that a wave motion is modeled by the partial differential equation $u_{tt} = v^2 u_{xx}$ whose solution is a function $u(x,t)$ of position and time. In the case of space of three dimensions, the wave function is $\psi(x,t)$, where $x = (x_1, x_2, x_3)$ a space variable.

For the atom of hydrogen (one proton and one electron) the attempt of **Schrödinger** to find a relativistic equation was not successful at first , because he did not know that the electron had a spin[52]. So after other considerations, he came with the equation

$$i\frac{h}{2\pi}\psi_t = H(\psi)$$

where H is d the Hamiltonian operator. Why exactly H? First **Hamilton** himself introduced his characteristic function in connection with the theory of ray optics, and this function has a link with the principle of least action. Second, unlike the Lagrangian, the Hamiltonian represents the total energy of a system which is needed in the modeling.

The presence of i in the left side means that $\psi(x,t)$ is itself a complex function : $\psi(x,t) = a + ib$. The problem is where did the imaginary i come from? We can see complex numbers in the solutions of differential equations like the $ODE : ay'' + by' + cy = 0$ which

[52] His resulted equation was known as **Klein-Gordon equation** which we will see later.

has two solutions of the form

$$e^{x(A+iB)} \quad \text{and} \quad e^{x(A-iB)}$$

The story with **Schrödinger equation** is that in place of momentum and position of a particle, we are dealing with time-varying quantum state $\psi(x,t)$. So the total energy $H = K + V$ where

$$K = -\frac{p^2}{2m}\Delta \qquad \text{and} \qquad V = V(x).$$

and this gives

$$H = -\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2} + V(x)\psi(x,t)$$

when it is subject to a potential V , the quantum state is presented by the equation

$$i\frac{h}{2\pi}\psi_t = -\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2} + V(\psi)\psi(x,t)$$

to every particle is associated a wave function $\psi(x,t)$. Δ is the ***Laplacian*** operator[53], it is also noted by the symbol or $\nabla^2 u$ or $\nabla \circ (\nabla \mathrm{u})$. It usually written for a function $u(x)$ as $\sum u_{xx}$. So $\frac{\partial \psi}{\partial x^2}$ is known also as $\Delta\psi$. If the particle is free(no potential) the equation is of the form

$$i\frac{h}{2\pi}\psi_t = -\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2}$$

whose solution is

$$\psi(x,t) = Ae^{-i(kx-\omega t)}$$

[53] Be careful! it is different from the difference notation : $\Delta x = x_2 - x_1$.

The angular speed is $\omega = \frac{hk^2}{m4\pi}$, and the solution at $t = 0$ may have the form

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int f(k) e^{-ikx} dk$$

The **Schrödinger** equation is one dimensional and *linear*[54] PDE , its right hand expression term is considered as a linear operator that generates eigenvalues. Its formal form is $\left(H - i\frac{h}{2\pi}\psi_t\right)\psi = 0$. When the equation is *stationary* , that's means that the quantum state does not depend on time t, it becomes

$$V(\psi) = -\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2}$$

A lot has been said about the importance of the **Schrödinger** equation. This is **Sommerfeld** who expresses his wonder

The most remarkable of all remarkable discoveries of the 20th century.

but how did **Schrödinger** get this miraculous equation ? **Feynman** once said on the **Schrödinger** equation

Where did we get that from? Nowhere. It is not possible to derive it from anything you know. It came out of the mind of Schrödinger, invented in his struggle to find an understanding of the experimental observation of the real world.

but other physicists do not agree. Schrödinger equation looks very much like the heat $u_t = u_{xx}$ and wave $u_{tt} = u_{xx}$ equations. The equation looks most like the **Hamilton-Jacobi** equation

[54] In fact it is parabolic as $b^2 - ac = 0$.

$$\frac{\partial S}{\partial t} = -H(\dot{p}, \frac{\partial S}{\partial p}, t)$$

we have seen before in analytic mechanics. The **Hamilton-Jacobi** equation is also seen as a wave equation although it is a nonlinear partial differential equation. The Schrödinger equation (stationary and time dependent) is seen as the eigenvalue problem. In the case $A\psi = \lambda\psi$, the operator A is the linear law which is:

$$\left[-\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2} + V(\psi)\psi(x,t)\right]\psi(x,t) = i\frac{h}{2\pi}\psi_t$$

The linear operator is the Hamilton H, it is a Hermitian operator then it is diagonalizable. And the time independent

$$\left[-\frac{h^2}{8\pi^2 m}\frac{\partial \psi}{\partial x^2}\right]\psi(x,t) = V(\psi)\psi(x,t)$$

So in the equation of the form

$$A\psi = E\psi$$

We can say that ψ is the eigenfunction and the resume:

−The the allowed energy levels E Are eigenvalue.

−The function ψ is eigenfunction , it represents the quantum state.

the solution is of the form :

$$\psi(x,t) = \sum c_n \Phi_n$$

where Φ_n are the eigenvectors.
There was intense conservations between physicists, everyone thought that the other

theory is wrong. **Schrödinger** showed also the equivalence of Heisenberg theory of matrices and his wave version. We have seen that in the case of the time independent equation we have

$$H\psi = E\psi$$

In 1926, three physicists: Dutch Hendrik **Kramers**$(1894-1952)$, German Gregor **Wentzel**$(1898-1978)$ and French Leon **Brillouin**$(1889-1969)$ developed a solution to linear equations including the Schrödinger equation. The method is known as the ***WKB approximation***. The stationary Schrödinger equation $H\psi = E\psi$ is a special case of the known Sturm-Liouville equation. Its solution can be obtained by using the Fourier separable variables method

$$\psi(x,t) = \varphi(x)\chi(t)$$

we get the solution $\chi(t) = Ae^{-i\omega t}$ and the general solution

$$\psi(x,t) = \varphi(x)e^{-i\omega t}$$

Introducing probability

But what is the meaning that we can give to the wave function $\psi(x_0,t_0)$? German Max **Born**$(1882-1970)$ in his published paper of 1926 proposed an astonishing interpretation that consist in the following :

The magnitude wave function the probability to find the a particle at the position x and at time t.

So

$$|\psi(x_0,t_0)| = \psi(x_0,t_0) \bullet \overline{\psi(x_0,t_0)}$$

thus

$$p = |\psi(x_0,t_0)|^2$$

The use of probability theory in physics was not new, it has been used in thermodynamics by **Maxwell** then by **Boltzmann**. The paper contained the *Born rule*

$$dp = |\psi|^2 dV$$

The whole of probabilities in space sum to unity

$$\int_V |\psi|^2 dV = 1$$

which is called *normalization*, $|\psi|^2$ here is a probability density function(PDF). dV is the volume : $dV = dxdydz$, it is also noted d^3r. So to locate an electron in space one has to find his function wave. To the real wave of **De Broglie** is substituted a *wave of probability*. In classical mechanics we can calculate the trajectory of a particle with all precision, but in quantum mechanics it is impossible, all what we can do is to use the **Schrödinger equation**, to calculate the probability that a particle is at the point x at the time t.

Schrödinger thought that electrons are probability waves orbiting around the atom, **Heisenberg** did not like the idea, he saw that they are not real waves but an abstraction.

The Schrödinger equation cannot model an electron with very big speed that is near to the speed of light c, so it is not a relativistic equation, we have to wait for Dirac equation. In 1927, after the discovery of spin, **Pauli** gave to an electron with spin ½ the equation

$$i\frac{h}{2\pi}\psi_t = \left[\frac{1}{2m}(\sigma * (p - qA)^2 + q\phi\right](\psi)$$

The symbol σ stands for three **Pauli matrices**

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The Copenhagen interpretation

Following wave mechanics and the introduction of **Schrödinger** equation, physicists began to try understanding this new world of physics, what was really going on and at the same time put their science on solid ground. They agreed that an electron being a particle can be also considered as a wave, and began to feel the necessity to figure out the meaning of the wave function, namely to give an *interpretation* for this new weird quantum mechanics. The problem is that the electron has two forms: a particle and a wave. But how can the electron be a particle and a wave at the same time? in his course **Feynman** says p. 10:

Historically, the electron, for example, was thought to behave like a particle, and then it was found that in many respects it behaved like a wave. So it really behaves like neither. Now we have given up. We say : " it is like neither ".

These new situation in physics was woven around **Bohr**$(1885-1962)$ in Copenhagen. Meanwhile **Heisenberg**$(18-194)$ was going to emit his uncertainty principle, he was also a main participant in the hot conservations with physicists like **Bohr** and **Born**. The physicists of the Copenhagen circle were supposed to give the interpretation of the duality of the electron. The strange result that would make a lot of controversy to this day was

The wave collapses to a particle when we make measure

According to this interpretation a particle has not a separate reality independent from the measurement we do. So the measurement causes the transformation

$$wave \longrightarrow electron$$

We will talking of a *chance* to find a particle at the place x and the tiame t. The quantum state of a particle at a place is probabilistic, we cannot predict that the electron can be at a certain place with $p = 1$. The new theory violates many well established principles of classical physics. In 1814 **Laplace**, author of the big "*Treatise of Celestial Mechanics*", talked[55] on determinism as a necessity, he saw the world as a mechanical clock, introducing his intelligent entity called later "Laplace demon" :

[55] In his book "*Essai Philosophique sur les Probabilities*" .

An intelligence which, for a given instant, would know all the forces by which nature is animated, and the respective situation of the beings which compose it, if moreover it were vast enough to submit these data to analysis, would embrace in the same formula the movements of the largest bodies in the universe and those of the lightest atom, nothing would be uncertain for it, and the future, like the past, would be present before its eyes.

Determinism of classical physics is replaced by randomness. This also caused the raise of the *measurement problem*.

Around the year 1925, **Einstein** and **Bohr** had intense discussion about the meaning of quantum theory. **Einstein** was very unsatisfied about the interpretation and even said that : " *God* does not pay dices".

Fig. **Bohr** and **Einstein** in 1925

The uncertainty principle

In his experiment thought known as *Heisenberg microscope*, **Heisenberg** introduced his principle that represents the random nature of quantum mechanics. The idea behind this is that the nature of the electron is very complex(particle and wave) and its speed is big, it is not imaginable how it exist around the nucleus.

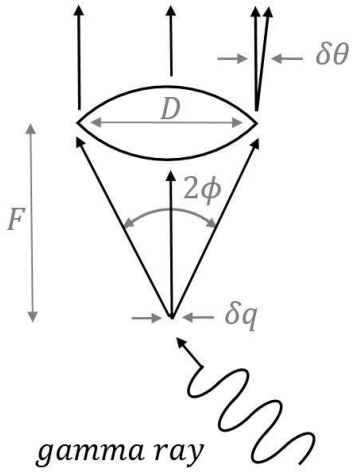


Fig. Sketch of the Heisenberg microscope

In 1927, from

$$MP - PM = i\frac{h}{2\pi}I$$

he found that

$$\Delta p \Delta x = \frac{h}{4\pi}$$

So the approximation depends on the very small number h . The principle sums to the fact that we cannot evaluate the speed and the place of a particle at the same time, so there is a bound of precision in a sense that

$$\Delta p \Delta x \geq \frac{h}{4\pi}$$

so if we know Δp very well, so our Δx is very bad such that their product can never be less than $\frac{h}{4\pi}$. The inequality is verified also for energy and time

$$\Delta E \Delta t \geq \frac{h}{4\pi}$$

With uncertainty principle classical physics is called ***determinist*** , with quantum mechanics we enter a new world of ***indeterminism*** with randomness motion and the calculation of probabilities.

The Bohr complementarity principle

Around the years $1927 - 28$, **Bohr** announced a principle on the duality nature of the quantum particles. It says that the wave-particle nature is necessary for a particle, and we cannot observe at the same time the electron as a wave and a particle.

Bohr:

In a situation where the wave aspect of a system is revealed; its wave aspect is concealed. Reveling both simultaneously is impossible. The wave and particle aspects are complementary

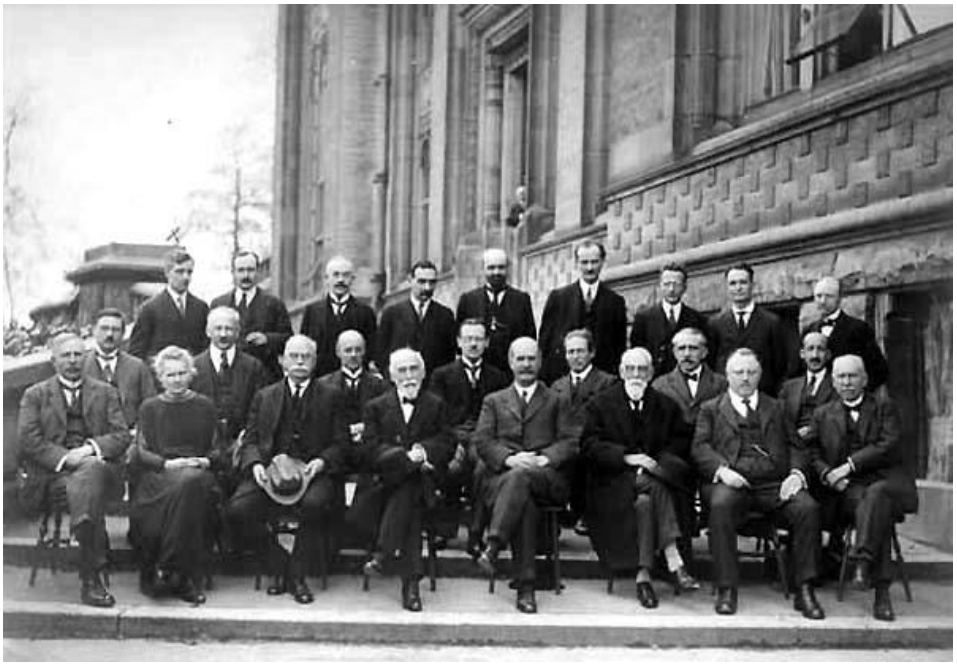


Fig. The fourth Solvay conference (1924).

Exercises

1) Show that the Schrödinger equation is linear.

1) let $\psi(x,t) = xte^{-it}$. Find : $\frac{\partial\psi}{\partial x\partial t}$, $\frac{\partial\psi}{\partial t^2}$, $\frac{\partial\psi}{\partial x^2}$. Show that : $\frac{\partial\psi}{\partial x\partial t} = \frac{\partial\psi}{\partial t\partial x}$.

1) find the de Broglie wavelength associated to a body of $m = 5g$ and speed $v = 12m/s$.

1) show that $p = \frac{h}{\lambda}$.

1) if a photon has a $1\ k\ eV$, find its momentum.

1)Calculate

$$\int_{-\infty}^{0} e^{x}dx \quad , \quad \int_{0}^{\infty} e^{-x}dx$$

1) find c in order that the following functions are PDF:

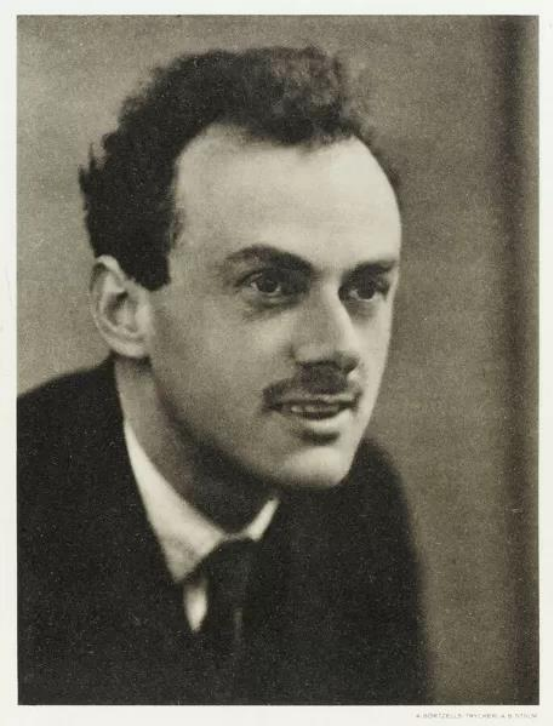
$$f(x) = \begin{cases} ce^{-x} & for\ x \geq 0 \\ 0 & for\ x < 0 \end{cases}$$

$$g(x) = \begin{cases} c\frac{x}{2} & for\ 1 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

$$g(x) = \begin{cases} c\sqrt{x} & for\ 2 \leq x \leq 5 \\ 0 & otherwise \end{cases}$$

1) Show that for free particle: $|\psi(x,t)| = |A|$.

QUANTUM FIELD THEORY



The discoveries made by Paul Dirac
had a much bigger impact
on the science of the 20th century
than those of Albert Einstein.

Antonino **Zichichi**

Paul **Dirac**$(1902-1984)$ was one of the greatest physicists of the preceding century and historians of physics say that through his three consecutive years, he shaped physics of the XX century. The years of his accomplishments were:

-1925: "*The fundamental equations of quantum mechanics"* founding quantum physics as a new theory.
-1926 : "*The Quantum Theory of the Emission and Absorption of Radiation",* founding the theory of quantum fields.
-1928: "*The Quantum Theory of the electron"* , founding the theory of elementary particles.

The scientific papers of **Dirac** were difficult to read by his fellow physicists. **Dirac** founded what we today call *quantum field theory(QFT)*. QFT is one the most important achievements in the history of physics. It is the theory of all particle physics that physicists have been doing in the XIX century and even today. **Dirac** is also known of his famous equation. Schrödinger equation was not relativistic, Dirac equation was relativistic, he made

a fusion between special relativity and quantum mechanics.

Origins

Charles **Dirac**$(1866-1936)$, the father of Paul **Dirac** was a Suisse from the French – speaking canton Valais. Charles left to England at the age of 20 and became a teacher of French there. His son Paul **Dirac** was born to an English mother, he got an education which showed that he aimed a mathematical oriented formation. Under the wish of his father, he entered the engineering school of Bristol in 1918 and received a bachelor degree in 1921. There, he studied in the department of mathematics for two years. Later he also received his PhD degree in physics.

Dirac was one the persons that introduced what we call a *generalized function*[56] . In fact in 1927, he introduced his celebrated function

$$\delta(x)=\begin{cases}0 & if\ x\neq 0\\ +\infty & if\ x=0\end{cases}$$

where strangely[57] $\int_{-\infty}^{+\infty}\delta(x)dx=1$. This not the sort of a function that mathematicians are used to . **Dirac** delta function has a relation with Heaviside function introduced by English Oliver **Heaviside** $(1850-1925)$

$$H(x)=\begin{cases}1 & if\ x\geq 0\\ 0 & if\ x<0\end{cases}$$

Such that we have $H'(x)=\delta(x)$. It is mathematically strange because in calculus if f is derivable then it is automatically continuous which means that a discontinuous function cannot have a derivative. In 1949, French Laurent **Schwartz**$(1915-2002)$ found a theory that could unify all these jottings and put them in a true mathematical theory. So these relations which are generalized function are now called *distributions*, a distribution is defined as a *linear form*. A *regular* distribution f is defined as

[56] Generalized function were introduced independently by other mathematicians as "weak solutions" to differential equations. Men like French Jean **Leray**$(1906-1998)$ in 1934, Russian Sergei **Sobolev**$(1908-1989)$ in 1933,…

[57] The same as probability normalization $\int_{-\infty}^{+\infty}f(x)dx=1$.

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\, \varphi(x) dx$$

The Dirac $\delta(x)$ is not regular. A distribution needs not to be a function, but all continuous functions are distributions. A distribution can be derivable even if it discontinuous, let f a discontinues function then

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle$$

where φ is derivable**. Dirac** also introduced the ***bracket notation*** in 1939. The notaion expresses the linearity of a state vector in quantum theory. The word bracket is divided into two parts :

- *bra* noted by the symbol $\langle \quad |$.
- *ket* noted by the symbol $| \quad \rangle$.

Both are just symbols for vectors. $|f\rangle$ is a vector that represents a quantum state. The aim is to present a quantum state, which means that is a particle is at a certain place. When two are in the of superposition we simply write : $|f\rangle = |f_1\rangle + |f_2\rangle$. This has the propriety of linearity $|a + b\,\rangle = |a\rangle + |b\rangle$ and even

$$|a + ib\,\rangle = |a\rangle + i|b\rangle$$

A vector A can be written in the form

$$A = a|i\rangle + b|j\rangle + c|k\rangle$$

A bra can act on a ket to give a scalar

$$\langle \quad | \longrightarrow | \quad \rangle \longrightarrow \langle \quad | \quad \rangle$$

to have a norm or a scalar. The operation is between a vector (a, b, c) and its conjugate $(\bar{a}, \bar{b}, \bar{c})$

$$\langle A|B\rangle = (abc)\begin{pmatrix}\bar{a}\\ \bar{b}\\ \bar{c}\end{pmatrix} = scalar$$

when $A = B$, it is just the norm. so in this new notation, Schrödinger equation is

$$ih|\psi_t\rangle = H|(\psi)\rangle$$

He also introduced the notation[58] $\hbar = \frac{h}{2\pi}$ in 1930.

A new idea

After that **Heisenberg** published his new ideas to quantum mechanics in 1925, **Dirac** was working on his thesis. His advisor English Ralph **Fowler**$(1889 - 1944)$ then recommended to **Dirac** to pursue the work on the new theory quantum mechanics, In 1926, **Dirac** completed his PhD thesis "Quantum Mechanics" containing his ideas.

Before that and in 1925, he published his paper "*The fundamental equations of quantum mechanics*" that founded his new view, and came to a generalization of the equation

$$(xy - yx) = i\frac{h}{2\pi}[x, p]$$

where $(xy - yx)$ is the Poisson bracket seen before in classical mechanics[59]. **Poisson** introduced these brackets in 1809 under the form

$$[x, y] = \sum_{i=1}^{n}\left(\frac{\partial x}{\partial q_i}\frac{\partial y}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial y}{\partial q_i}\right)$$

[58] It is perhaps $\hbar$, the 23rd Serbian letter.

[59] Dirac found the idea of Poisson brackets in "*Analytical Dynamics…*", a book by British scientist Edmund **Whittaker**$(1873 - 1956)$.

they are used in Hamilton mechanics too. The equation is already written by **Born** and **Jordan** $AB - BA = \frac{h}{2\pi i} I$, but without Poisson bracket.

Quantum theory of fields

In 1927, he published his paper "*The Quantum theory of Emission and Absorption of Radiation*" founding the great idea of *quantum field*. It was in this paper that he introduced his famous generalized function $\delta(x)$.

It is known that in electrodynamics we have two fields : electric and magnetic. And the two phenomena of that repulse or attract as lines of forces. A particle is seen as a wave and at the same time in the new theory of **Dirac** as a *field*, so there is a strong between a particle and the field: an electron is the perturbation of a field : the electron field. So the new idea is formulated as follows:

> Every particle(and consequently a wave) is an excitation of a field, called a *quantum field*, defined on all space and time.

Dirac grandiose idea is that a force as an interaction is carried by particles. So a particle can be a *force carrier* or *quantum of the force field* . He advanced the idea that there is a quantum electric field around the particle. So we talk about electron field, a proton field, a neutron field,… The idea would be generalized this quantum field theory to all forces. In modern QFT we have:

– The photon is the carrier of electromagnetic force, it is the quantum of electromagnetic field.

– W and Z bosons are the carriers of electroweak force , it is the quantum of electroweak field.

– The gluon is the carrier of strong force , it is the quantum of strong field.

– Higgs boson is the carrier of the Higgs field.

QFD is the theory used in all physical theory since **Dirac** in the huge subject of particle physics, black holes,…
Quantum field theory has succeeded to explain all particles and forces(interactions) except gravity.

The theories we will see later are all in the realm of QFT :

- Quantum electrodynamics(QED).
- Quantum chromodynamics(QCD).
- Quantum flavordynamics(QFD).
- Gauge theory.

The Dirac equation

Dirac published in 1928 "*The quantum theory of the electron*" , his greatest contribution to quantum theory, he could explain the movement of electrons in any speed, it is a relativistic equation unlike Schrödinger equation which is non-relativistic. He read Heisenberg paper of 1925 and he could his method to and that Heisenberg can be formulated by matrices.

Dirac mechanics uses the Lagrangian, he introduced :

−The c −numbers.

−The q −numbers

which we will see later. The **Dirac** equation is like that of Schrödinger, its aim is to explain the electron movement.

$$ih\psi_t = \left(\beta mc^2 + c\sum \alpha_n p_n\right)\psi(x,t)$$

or

$$\left(ih\psi_t - \beta mc^2 - c\sum \alpha_n p_n\right)\psi(x,t) = 0$$

the $(\beta mc^2 + c\sum_1^3 \alpha_n p_n)$is the energy , we note the presence of βmc^2 from special relativity and momentum p_n. The sum from 1 to 3 is for three dimensional space. This is a fusion of the theory of special relativity and quantum mechanics. The wavefunctions of wave mechanics are considered as *fields*

$$\psi(x,t) \coloneqq field$$

The Klein-Gordan equation

Another equation similar to **Dirac** equation is

$$\frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} + \frac{m^2c^2}{\hbar^2}\psi(x,t) = 0$$

usually written in the form $(\square + \frac{m^2c^2}{\hbar^2})\psi(x,t) = 0$, where, as before, $\square \coloneqq \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the d'Alembert operator. It was proposed in 1926 by Swedish Oskar **Klein**$(1874-1977)$ and German Walter **Gordon**$(1893-1937)$. The equation is the right , but it models only particle without *spin*. It has been found by **Schrödinger** earlier and he did not publish it. Special relativity says that a particle is mass and energy. Remark that when $m = 0$ and spin $s = 1$ we have the photon equation i.e. the Maxwell wave equation $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ which is relativistic.

There is also another related equation due to Romanian physicist Alexandru **Proca**$(1897-1955)$ namely

$$\partial_\mu(\partial^\nu B^\nu - \partial^\nu B^\nu) + \left(\frac{mc}{\hbar}\right)B^\nu = 0$$

There is also the Weyl equation due to German mathematician Hermann **Weyl**$(1885-1955$.

The discovery of antimatter

In 1928, in the same paper, **Dirac** discovered a very new phenomenon. So when one combine the two theories he found two solution of his equation $(ih\psi_t -)\psi(x,t) = 0$, and the energy solution are f the form

$$E_1 = c\sqrt{p^2 + m^2c^4} \ldots \qquad \text{and} \qquad E_2 = -c\sqrt{p^2 + m^2c^4} \ldots$$

the first is positive but the other is negative(how energy can be negative!). To explain this, **Dirac** proposed his "*Dirac sea*" that led to a totally new ting : the *antimatter*. This means that the two solutions are opposite in nature

$$matter \leftrightarrow antimatter$$

The antimatter was first noticed in 1932 by American Carl **Anderson** $(1905 - 1991)$ in cosmic rays.

In the case of electron particle, the antiparticle is the *positron* . The antiparticle of a particle can be different by charge and quantum numbers.

Particle	***Antiparticle***
electron e^-	positron e^+
proton p	antiproton $\bar{p}$
neutron n	neutron $\bar{n}$
neutrino ν	antineutrino $\overline{\nu}$
Higgs boson H^0	Higgs boson $\overline{H^0}$?
quark q	antiquark $\bar{q}$
meson $(\pi^+, \ldots)$	antimeson$(\pi^-, \ldots)$
graviton G	?

Fig. Particles and antiparticles

The positron has the charge of $1.59 \bullet 10^{-19}\ Coulomb$. What happens to a electron and a positron? they annihilate each other to give energy

$$e^- + e^+ \longrightarrow \gamma$$

The Dirac theory does not contradict the work of physicists before him, it generalizes it. We return now to c −numbers and q −numbers.

− c-numbers : The letter c stands for *classical*. They stand for classical quantities that is commutative numbers, as real numbers $\mathbb{R}$ and more general complex numbers$\mathbb{C}$. The eigenvalues of the matrix are the c −numbers.

− q-numbers : The letter q stands for *quantum*. Quantum quantities q-numbers are operators, they are in the place of operators, so the matrices of **Heisenberg** are q −numbers. Thus, q −numbers can be not commutative.

The book of von Neumann

In 1930, **Dirac** published his book "*The Principles of Quantum Mechanics*" which was a masterpiece, a classic of quantum theory. In 1932, Hungarian John **von Neumann**$(1903 - 1957)$ published a book in German language, following the mathematical work done by Dirac, to treat quantum mechanics in a " mathematically rigorous manner". The book was translated into English under the title : "*Mathematical Foundation of Quantum Mechanics*". **Dirac** has already worked the idea with his delta "function" $\delta(x)$ but the mathematical soundness of this function waited until the introduction of the theory of distributions in the $1940's$.

Von **Neumann** methodology was to put quantum theory on a firm ground and optioned in this case to use a theory with *axioms*, thus he was influenced by the methods of his teacher David **Hilbert**. The idea of is to use objects called Hilbert spaces.

We have already seen linear spaces. A Hilbert space $\mathbb{H}$ is a special linear space. Unlike usual space $\mathbb{R}^n$, a Hilbert space can have infinitely many dimensions. Our usual space $\mathbb{R}^3$ itself is a Hilbert space. The particularity with the Hilbert space is that we can define useful proprieties like : a vector v_1 is *perpendicular* (or orthogonal) to another vector v_2 . we can also define the angle between two vectors. We know that two vectors are perpendicular if $v_1 \circ v_2 = 0$. To do this the norm (length) of an element in $\mathbb{H}$ is not like in other linear spaces, it comes from a scalar product. We have seen scalar product, here we use a generalization of it called *inner product*, by which we can introduce complex numbers

$$\langle x|y\rangle : \mathbb{H} \times \mathbb{H} \longrightarrow E$$

so the norm is linked to the inner product $\|x\| = \sqrt{\langle x|x\rangle}$. A norm in linear algebra comes from inner product if and only if it satisfies the parallelogram law

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$$

The *spectrum* of an operator in a Hilbert space is the set of its eigenvalues. We know that in finite spaces, to every linear operator is associated to a matrix. The Hamiltonian is a linear operator and can be represented by a matrix.

A vector space like a Hilbert space can be endowed with an *orthonormal* basis with the proprieties:

– *ortho* form orthogonal (perpendicular) , all the vector of the basis are orthogonal.

– *normal*(lenght equals to 1) we say that a vector v is normal if its module is 1. If we have a vector v we can normalize it, i.e. get $\|v\| = 1$, by making a division on its magnitude

$$\frac{v}{\|v\|}$$

So we do a *normalization* , we say that we normalize the length to unity.

It is called a *Hilbert basis*[60]. We know that the Heisenberg matrices are linear operators and they represent the proprieties of a quantum systems: momentum, spin, ….

–*Observer*: it is the entity that makes the measurement of a system, like an apparatus. The particle which we ignore trajectory and speed, energy, position have quantum states. These states are points in a set with a certain structure. The set is called a *Hilbert space*. So a Hilbert is not composed of particles, but of particles quantum states.

[60] There is a method in linear algebra called by ***Gram-Schmidt*** *method* that permits to transforms a set of vectors to an orthonormal basis.

−*Observable*: it is the proprieties of a quantum system that we observe, that is we can measure like : position, momentum, …the situation of a system is an eigenvalue problem $AX = \lambda X$, and an observable is represented by an operator, it is usually a Hermitian operator (like a matrix). In this theory there are non-commuting observables.

So we can give a glimpse of the reformulation:

The measurable quantity (like position,…) is an Hermitian operator A acting on a state space (Hilbert space $\mathbb{H}$). The components of $\mathbb{H}$ are vector states. The eigenvectors of A are the basis of $\mathbb{H}$. The measured quantities we interested in are the eigenvalues of A . The eigenvalues of the operator can for example be the energy levels.

So the proprieties of a particle are considered as operators on a Hilbert space that represent the systems. **Dirac** and **Schrödinger** both showed that wave mechanics and matrix mechanics were equivalent. **von Neumann** proved this by considering the isomorphism between two sets : the space of sequences a_n such that $\sum^{+\infty}|a_n| < +\infty$, and the space of functions f such that $\int_a^b|f(x)|dx < +\infty$, which are both Hilbert spaces.

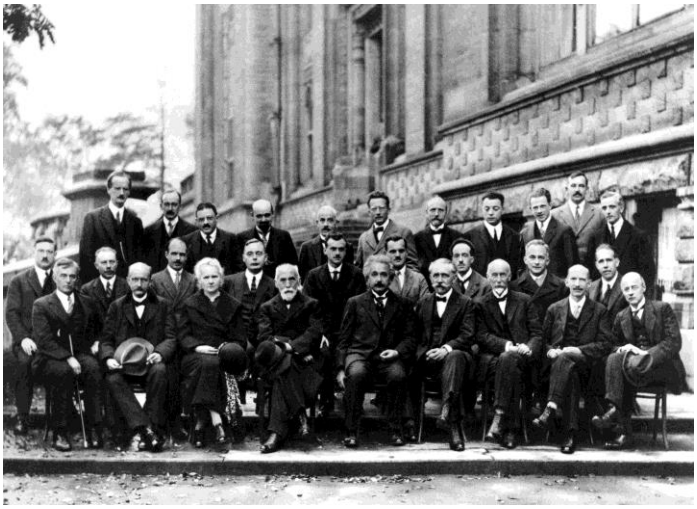


Fig. The fifth Solvay conference (1927).

Exercises

1) graph the following two functions

$$f(x) = \begin{cases} 1 + x & if \ -1 \le x < 0 \\ 1 - x & if \quad 0 \le x \le 1 \end{cases} \qquad g(x) = \begin{cases} x^2 & if \ -1 \le x < 1 \\ x & if \ 1 \le x \le 2 \end{cases}$$

1) show that $|a + ib\rangle = |a\rangle + i|b\rangle$

1) Show that: $\|x\| = \sqrt{\langle x, x\rangle}$

1) show that : $\langle v|v\rangle = \|v\|$.

1) show that : $p = \hbar k$.

1) Normalize the following vectors:

i) $(1,2,3)$

ii) $(-1,-2,-3)$

iii) $(0,0,1)$

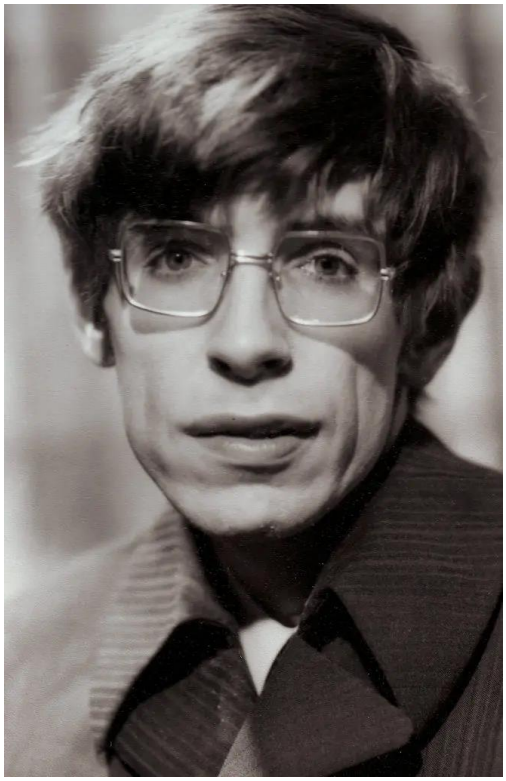
1) show that the spectrum of $\begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$ is real.

1) show that : $\|x + y\| + \|x - y\| \le 2\|x\| + 2\|y\|$

1) show that : $\|x + y\| + \|x - y\| \ge 2\|x\| - 2\|y\|$

1)Show that the norm of $\frac{v}{\|v\|}$ is 1.

THE SEARCH FOR UNITY



…These two theories are known to be inconsistent with each other—they cannot both be correct. One of the major endeavors in physics today, and the major theme of this book is the search for a new theory that will corporate them both—a quantum theory of gravity. We do not yet have such a theory, and we may still be long way from having one, but we do already know many of the proprieties that it must have.

*Stephen **Hawking***

The idea of unifying forces was not new at the time of the creation of quantum theory in 1900. Before the XX century, scientists knew about unification of theories in many branches of science[61], and we have in the realm of physics two very known examples:

–**Newton** has already unified the force responsible of falling bodies on earth and the force of gravity existing between planets in one theory of universal gravity and presented it in one equation.
–Then came **Maxwell** with his unification between electricity and magnetism in the theory of electromagnetism with 4 equations.

No problem faced physicists in the XX century like the problem of gravity force. Quantum mechanics is a true theory, it works well. We believe in it and the proof of this is that it has been very successful in performing a lot of things. In fact its theories helped people to do wonderful inventions. For example it helped to invent very useful technologies like masers, lasers, transistors, powerful microscopes,… although gravity was explored first by **Newton** then by general relativity , we still do not understand it well as we do with the other forces

[61] There have been many unifications in mathematics, in biology

of nature. We still ignore the mediator of the force and also we do not know what relation it has with quantum mechanics.

Why unify?

After the founding of classical mechanics and electrodynamics, physicists were almost sure that the only forces in nature are gravity and electromagnetic forces, which are responsible of many events in nature. But with the advent of quantum theory, it turned out that there exist 2 other forces which make the total of *4 forces*. The new 2 forces and these forces work inside the atom and like the two old ones, are responsible of the existence of our universe. Physicists began to think what the origin of the forces was. First the forces are very different, so to think to unify them, is to that they have the same origin force, which is a bit challenging. The idea can emerge easily from the following observation:

–Two electrons repulse each other by the electromagnetic force and at the same time they attract each other by the gravity force.

–An electron and a proton attract each other by the electromagnetic force but at the same time they attract each other to gravity force.

So in both cases, is it possible that the two forces, gravity and electromagnetic, can be unified in one force, a law that can play the two roles. Can we with one force that? Perhaps, we cannot rule that out because

After these exploits, the new possible tasks to be achieved in the future are:

– unification of quantum mechanics with gravity.
– unification of weak and electromagnetic forces.
– unification of weak and strong forces.
– unification of weak and gravity forces.
– unification of strong and gravity forces.
– unification of strong and electromagnetic forces.
– unification of gravity and electromagnetic forces.

we have seen **Newton** and **Maxwell** succeed some unification, and after them **Dirac** succeeded in unifying quantum mechanics with special relativity. But physicists dreamt of more than that, and the goal during the XX century was to unify all the forces in one force.

We have seen that **Weyl** made an unsuccessful attempt to unify gravity and electromagnetism. **Einstein** spent a lot of time until his death in 1955 to get a unified theory by considering electromagnetism in the frame of space-time. We still lack a quantum theory of gravity.

The four forces

The 4 forces discovered by physicists are also called *interactions*, particles interact to each other through the forces. These three forces (the four except gravity) with their particles constitute what is known as the *standard model*. The four forces of nature are different by :

- Range the distance the force acts. there are short-range forces and long-range forces.
- Nature of the force ,
- strength, So order of the forces according to their strength :

$$strong > electromagnetic > weak > gravitational$$

- particle on with they affect the force.
- the mechanism,

We will expose them with some explanation:

- *The gravity force (between masses):* the first force, it is known prior to **Newton**, it is very weak and any two with masses m_1, m_2 can interact, and the law can explicitly by the formula $F = G\frac{m_1 m_2}{r^2}$. It is believed that the particle that is responsible of carrying the force is *graviton* : it has never been found. Its range is unlimited. Gravity holds the planets in orbits around the Sun. Without gravity there would be no universe, no life. General relativity gives the law of gravity in the universe.

- *The electromagnetic force(between charges):* it is the force that exists between charges only, it is repulsive between same charges and attractive in the other case. It is computed by the coulomb law $F = k\frac{q_1 q_2}{r^2}$. Its range is unlimited. It is responsible for the binding of electrons to nuclei, for the composition of molecules and organic life. The particle that is responsible of carrying the force is the *photon*.

–*The strong force (between colors):* when the structure of the atom nucleus has been discovered, physicists were wondering how protons of the same charge bind to each other inside the nucleus without repulsion. So, a new mysterious force that binds protons together against the force of repulsion should be exist(it is also believed that neutrons soften the repulsion). In fact this repulsion can explain why heavy atoms like the Uranium atom are radioactive. The mechanism of strong force was first explained by Japanese Hideki **Yukawa**$(1907-1981)$, who stipulated the carrying force : the *meson*. When bombarded, the atom can break down and protons , the energy is liberated as heat according to the mass-energy formula

$$E_2 - E_1 = (m_2 - m_1)c^2$$

so, the strong force in fission breaks down. Unlike fission, the fusion of protons in one atoms needs a lot of energy, fusion is stronger and implies a lot of energy.

The strong force is short-range force. The strong force is also responsible for the bind of quarks together inside protons and neutrons. It acts between neutrons, between protons and between neutrons and protons. the particle that is responsible of carrying the force is the *gluon*.

–*The weak force(between flavors):* the mechanism of this force is not very clear as the precedent, it causes some decays of the nucleus. It is a nuclear force, it explains radioactivity, its biggest source of energy is the sun. It has been discovered through the two known disintegrations: *beta* β^-, in which a neutron transforms into proton

$$n \longrightarrow p + e^- + \nu_e$$

it makes a proton, an electron and a neutrino from a neutron, they seem to be tied together before then. And *beta* β^+: in which a proton transforms into a neutron

$$p \longrightarrow n + e^+ + \bar{\nu}_e$$

it makes a neutron , an positron and a neutrino from a proton, and as before they seem to be tied together before then. It was first explained by Italian Enrico **Fermi** in 1933. It is a

very short range force. In fact, It is a weak force because it is short-range. It plays a role in the generation of energy inside the sun. This is accomplished by the transformation

$$H + n \longrightarrow D$$

to give deuterium atom D which is also noted ${}^{2}H$. Then the fusion reaction between two deuterium atoms that gives helium and produces energy

$$D + D \longrightarrow He + n$$

also it is presents in fission. Unlike fission, fusion is a hard operation to control but can be liberated in thermonuclear weapons (hydrogen bomb).

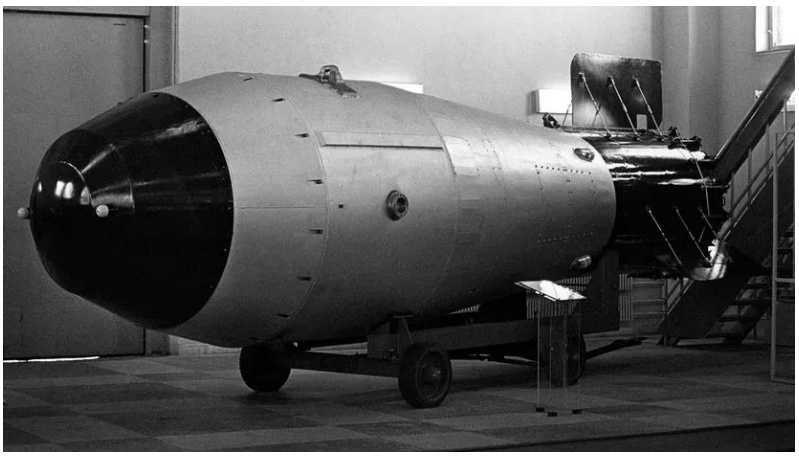


Fig. The soviet “Tsar Bomba”,
the most devastating hydrogen bomb ever fabricated.

The weak interaction is thus responsible in beta decay. In fact it transforms a quark to another, or otherwise it transforms a flavor to another flavor. the particle that is responsible of carrying the force are the W^{+}, W^{-}, Z^{0} bosons.

force	particle	strength	range	Intermediate vectors
strong	quarks	1	$10^{-15}m$	gluon
electromagnetic	charges	2	unlimited	photon
weak	Leptons and quarks	3	$10^{-17}m$	W^+, W^-, Z^0
gravity	matter	4	unlimited	graviton

Fig. the four forces with characteristics.

The unifying theories

Physicists have proposed many theories and ideas to unify two forces or more of the 4 known forces. Those theories are known under the name of *unifying theories*. After we have seen the idea of symmetry and gauge theory, we still wonder where all this can be leading to, namely physicists themselves do not know if they really on the right way. So we can cite :

–The grand unified theory (GUT) aims in general to unify three forces : weak, strong and electromagnetism. Some say that perhaps these three forces were one force at the early formation of the universe. It was first proposed by American Howard **Georgi**$(1947-)$ in 1974.

– The theory of unification of gravity and electromagnetism. Began by German Hermann **Weyl**$(1885-1955)$ in 1918, but failed. The idea has seen the birth of gauge theory. Then another attempt by German Theodor **Kaluza**$(1885-1954)$ in 1919 and Oskar **Klein**$(1894-1977)$ which needed a 5 dimensional space.

–String theory.

–Loop gravity theory.

–A big far reaching dream is the unification for all four forces in one theory, It is known among physicists under the name of the *theory of everything*. Next to unify the four forces,

there is another smaller task that remained is the unification between quantum mechanics and general relativity in one system.

The electro-weak force

the unification efforts began in 1961, when American Heldon **Glashow**$(1932-)$ could the weak interaction in $SU(2) \times U(1)$. $U(1)$ is the group of complex numbers z such that $|z| = 1$. It is considered as the group of rotation in the plan:

$$U(1) = \{e^{i\alpha}\}$$

where α is real. $SU(2)$ is also a group of rotations more complicated than $U(1)$. In 1968, it was Pakistani Abdus **Salam**$(1926 - 1996)$, and Steven **Weinberg**$(1933 - 2021)$ who succeed to make the unification of the two forces : weak and electromagnetic force to become simply the *electro-weak*. The electromagnetic forces succeeds the weak force

$$electromagnetic > weak$$

the three physicists succeed to make the unification by using laws of symmetry and conservation. The ingredients of symmetry were the gauge group $SU(2) \times SU(1)$. **Salam** began the work in 1959, the tools are special unitary groups like $SU(3)$ and also on *normalization* theory to avoid infinities. The idea turns around *perturbation theory* which is the theory of approximation of a complicated quantum system by a simpler one. The important gauge symmetry breaking played a crucial role in their theory, it was discovered that left-right symmetry was violated in weak interaction, but conserved in electromagnetic integration. **Salam** also contributed to the theory of Higgs boson. In fact he and **Weinberg** began in 1964 to adjust the Higgs mechanism to the electroweak symmetry. The force manifests itself between particles in a phenomenon called *neutral weak currents*. They predicted that the new force are transmitted by particles

$$W^+, \quad W^-, \quad Z^0$$

which are bosons and were discovered in 1973 and 1983. The W^+ boson has positive charge and is the antiparticle of W^- boson which has a negative charge. The Z^0 boson has no charge and it is its own antiparticle. The three particles are really massive, they are

heavier than the protons and neutrons!!! Among these, we have the mechanism first beta decay (weak force)

$$n \longrightarrow p + W$$

Then the intermediate W

$$W \longrightarrow e^- + \bar{\nu}_e$$

and

$$Z^0 \longrightarrow e^- + e^+$$

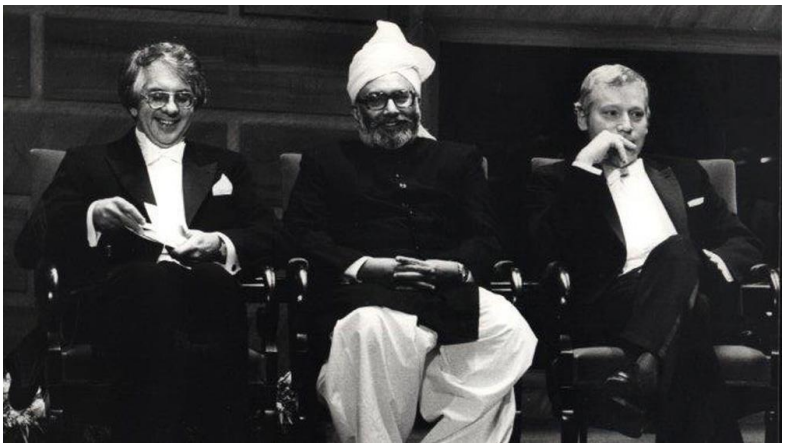


Fig. The three men responsible of

the first unification in terms of gauge theory

unlike gravity and electromagnetic forces, the weak force violates parity. The electroweak synthesis shows very well the power of symmetry and conservation.

Exercises

1) How much is electromagnetic force smaller than the strong force?
1) How much is weak force smaller than the electromagnetic force?
1) How much is gravity force smaller than the electromagnetic force?
1) How much is gravity force smaller than the weak force?
1) How much is gravity force smaller than the strong force?
1) How much is weak force smaller than the strong force?

1)calculate the electromagnetic force between two charges in the following cases:
i) $q_1 = 5\ C$ and $q_1 = 7\ C$ separated by 20 cm.
ii) $q_1 = 50\ C$ and $q_1 = 70\ C$ separated by 10 m.
iii) $q_1 = 5\ C$ and $q_1 = 5\ C$ separated by 15 km.

1)calculate the electromagnetic force between two electrons.
1)calculate the electromagnetic force between two protons.
1)calculate the electromagnetic force between an electron and a proton.
1)calculate the electromagnetic force between an electron and a neutron.
1)calculate the electromagnetic force between two neutrons.

1)calculate the gravitational force between two masses in the following cases:
i) $m_1 = 5\ g$ and $m_2 = 15\ g$ separated by 10 cm.
ii) $m_1 = 50\ g$ and $m_2 = 500\ g$ separated by 10 m.
iii) $m_1 = 15\ g$ and $m_2 = 750\ g$ separated by 10 km.

STRING THEORY

Some string theorists prefer to believe that string theory is too arcane to be understood by human beings, rather than consider the possibility that it might just be wrong

Lee ***Smolin***

As we have seen, after that **Maxwell** unified electricity with magnetism in what is known as electrodynamics, and **Dirac** unified quantum mechanics with special relativity,… the task of unifying the four forces of the nature has been a big dream of physicists across the XX century and it continues until our days. Theoretical physicists wanted to finish the job and unify the common laws, and of course, this unification could help to understand the constitution of the universe and explain how it began in the far past. So physicists have proposed new theories in hope to find the final theory, that which these theories like string theory and loop gravity theory are still persisting.

The unification of nature forces is considered as the holy grail of physics, the ultimate knowledge. So we always hear the name: *Theory of Everything* , which means that there is (perhaps) a law, let say an equation (or a set of equations) that can encompass all the forces, like the four **Maxwell** equations that unified electricity and magnetism. This equation is often called the God equation[62],… so does this equation exist? Physicists do not know really.

In the stream of the theory of everything stands *string theory*. But string theory aim is only to give gravity a quantum flavor. And some hope that this theory of strings might be the candidate for the theory of everything.

[62] From the book of physicist Michio **Kaku**$(1947-)$: *The God Equation*(2021).

The problem with gravity

After the creation of the two theories: quantum mechanics and general relativity, what remained for physicists was to know how gravity really worked. It has eluded every attempt to quantification. So physicists wanted to understand gravity as a quantum theory, or to turn it into a quantum theory.

To quantize gravity, means according to this point of view, physicists have to gravity in quantum field theory as the other three forces with which the associated particle is the graviton. But there are many problems facing them:

–First the gravity force is very weak force, it is the weakest of four forces. And its associated conjectured graviton turned to be very difficult to detect.

–unlike the other forces, gravity is common to all particles.

– gravity works at a big scale.

But this will not solve the problem of unification, because to the everything theory all we have to fusion the other force together to find a sole law that governs. So until now we could not made a fusion between strong and electromagnetism, or between gravity and weak forces….. Can string theory succeed in explaining gravity ? will string theory be the theory of everything?

Beginning

In the $1960's$, physicists with the mechanism strong force that bind together protons in the nucleus, so it was a theory for hadrons and hadronic scattering. Scattering here means that the atom absorbs a photon and then emits another. The Regge trajectory of a proton in function of its angular momentum developed by Italian Tullio **Regge**$(1931-2014)$. This linear Regge trajectories could not be derived by QFT. The theory began when in 1968 Italian physicist Gabriele **Veneziano**$(1942-)$ his appear : "*Construction of a crossing-symmetry ,Regge behaved amplitude for linearly rising trajectories*". They were studying hadronic scattering by a matrix called. He found that that the function

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

has some relation to describe particles strong force(proton,…) as a *probability amplitude* by giving a formula for that.

$$constant \bullet [B(a(t), a(x)) + B(a(t), a(r)) + B(a(r), a(x))]$$

This function in a form of a definite integral, is known as **Euler** beta function the other is the gamma function defined as

$$\Gamma(x) = \int_0^{+\infty} e^{-t}\, t^{x-1} dt$$

and it a very important in many like analysis , number theory (zeta function) and even in physics. We know that $\Gamma(n) = (n-1)!$ and we have a relation between them

$$B(x, y) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(x+y)}$$

So to gravity theory as a quantum,

Strings

It was Japanese Yoichiro **Nambu**$(19 - 2015)$ and others who in the early 1970 proposed that can be as a vibrating string. In 1960 the idea string in place of a particle began to be adopted to the strong force inside the nucleus.

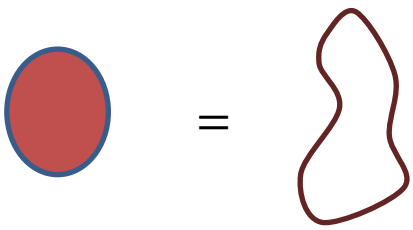


Fig. a particle is seen as string

Strings have never been seen like seen particles. They are considered as filaments of energy vibrating, when a string vibrates it become particle according to it is vibration. is that a particle being a wave is also a " string".

Fig. closed string and open string.

So a particle is just a string vibrating in a certain way.

Fig. two strings give another

Many of its : John **Schwartz**, Brian **Green**, continue to support the theory. Mathematically a string is vibrating at the distance x and the time t by a harmonic given by **Bernoulli** as

$$y(x,t) = sin(2x)cos(2t)$$

Something between zeta and string theory which is false mathematically[63] like:

$$\zeta(1) = \sum_{1}^{+\infty} \frac{1}{n^{-1}} = -\frac{1}{12}$$

[63] It looks like the strangeness of the Dirac function that equal to $+\infty$ at $x = 0$.

There are a lot of ideas and speculations around the theory of strings. But string theorists has ended to three ideas : unification, extra dimensions, supersymmetry.

Bosonic string theory

String theory is not primarily a gauge theory. It started as a theory for bosonic particles, it is to bosons not hadrons because massless particles like photon cannot be for hadron. Although the idea first dedicated for strong force it has then been abandoned for QFD which has been very successful with the QCD by **Gellman**. Japanese Yoichiro **Nambu**$(19-2015)$ has in the 1970 proposed other formulas as a string that has a length l and , the action of a string is given by the formula

$$S = -T \int_{t_1}^{t_2} dt \int_{0}^{l} \sqrt{-\gamma} d\sigma$$

But the idea of remained for peculations until the work of American John **Schwarz**$(1941-)$ in 1984 with superstring theory. The is on the work of Russian Alex **Polyakov**$(1945-)$ with his action.

$$S = -\frac{T}{2} \int h^{\alpha\beta} g_{\alpha\beta} \sqrt{-h} dt d\sigma$$

the space-time is of 26-dimentional.

Problem of dimension

The ideal of dimension more than 3 is not strange, it has been introduced first in geometry. Before string the idea of higher dimensions was the **Kaluza-Klein** theory in the $1920s$. In 1919, German Theodor **Kaluza**$(1885-1954)$ proposed a theory to unify gravitation and electromagnetism , and he came with the idea and the fifth dimension. Later, Swedish Oskar **Klein**$(1894-1977)$ gave an explanation to the theory in terms of quantum theory.

What is the complicated the things was the appearance of multidimensions. This had the nasty effect that its adherents abandoned the theory until the $1980's$. With one time t so the 11 dimension called M-theory .

string theory does not only predicts filaments but, predicts also surfaces called *branes*, the idea was introduced the word in 1987 contraction of "*membrane*" by . so $p-$ brane is with p spatial dimensions

$-$ 0-brane : point particle
$-$ 1-brane : string
$-$ 2-brane : also known as $D-$brane, strings can linked by $D-$branes. Quantum fields described by Yang-Milles theory involves strings that are attached by D-branes.

Linked to the dimensions in string theory, there is a principle known as the holographic principle.

$-$ The *holographic principle* : Holography is a technique invented by Hungarian Dennis **Gabor**$(1900-1979)$ in 1948 and the idea is to construct an image in 3 dimension by the recording the wave phase with the interference of laser light . Holography is from two Greek words :$o\lambda o\varsigma$ the whole and $\gamma\rho\alpha\phi\varepsilon\iota\nu$ to write(or to draw). The holographic principle came from the study of black holes, to be a foundation that may give credence to string theory, it has proposed by Dutch Gerard **'t Hooft**$(1946-)$ it says that the description of a volume of 3-dimesion space can be seen as another in a region of lower dimension namely the boundary.

Superstring theory

In $1970's$,a new super-symmetry theory by French Jean **Gervais**$(1936-)$and others saw light. This new development lead in the year 1984, to a new string theory that encompasses also fermions called *super-symmetry string theory* or *superstring theory* launched by of American John **Schwarz**$(1941-)$ and revived the theory. This time, the spacetime is of 10-dimentional.

In place of QFT , **Heisenberg** proposed in 1942 his S-matrix theory to deal with quantum field theory, The letter S is for scattering, it is a matrix that is unitary matrix$(AA^{*}=I)$ and preserves. At first, physicists proposed it again as theory to treat strong force in the 1960 and abandoned QFT. But what happened then is that the former was abandoned another time in 1970 after the success of QFT.
Another development of string theory was M-theory, it is a theory with 11-dimentional space-time. It has conjecture by American Edward **Witten**$(1951-)$ in 1995, as a unique theory for the existing string theory. And a work by American Leonhard **Susskind**$(1951-)$ then afterwards string theory adopted the S-matrix was introduced to explain how the extra dimensions can be reduced to 4 only.

Black holes

It is believed that string theory, if it is a true theory, can find solutions to many problems about *black holes*. A black hole is a mysterious dense place where the force of gravity is very strong , and from which nothing can escape. The existence of black holes was predicted by the theory of general relativity as we have seen. When the **Einstein** equations break at a singularity, the curvature is not finite, and there the equations break down and the solution becomes infinite. So a black hole is seen as the singularity solution of the field equations, this was not very understood in the $1920's$. The name "black hole" was coined in the 1960's.

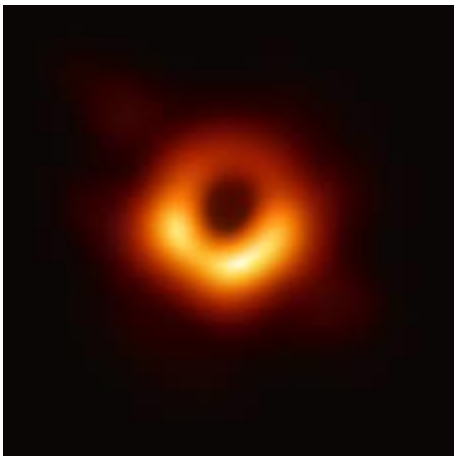


Fig. A black hole as photographed in 2019.

And where is the problem?

It is not known[64] if string theory is really a correct theory that can deal with the problems sent to at the first time. The same is with the theory of loop quantum gravity we will see later. The older string theory is still a very controversy theory. Among its specialists and lovers themselves we can read expressions like

" not an empirically well-confirmed theory",

64 See for example Lee **Smolin**$(1955-)$: *The Trouble with Physics(2006)*. The French edition is foreworded by mathematician Alain **Connes**. Also mathematician Peter **Woit**$(1957-)$ wrote : *Not Even Wrong(2006).* Another book is : *Farewell to Reality…* written by Jim **Baggott**$(1957-)$.

" much effort has been put into string theory in the last twenty years, but we still do not know whether it is true",…

Although there is a lot of mathematics around, there are no really experimental tools to prove the consequences of the theory. Unlike quantum mechanics and its affiliations like $QFD, QED, QCE\ QFD$, string theory is a mathematical theory without physics. The strings that are supposed to be the bricks of matter are so small, of the order of the ***Planck length*** :

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 10^{-33}\ cm$$

and it is estimated that an accelerator of the size of our Galaxy is needed for the strings to be observed !!!!

so it is really a physical theory yet without experiment to corroborate its soundness and predictions. String theory even uses the tools of knot theory, a very difficult mathematical theory…

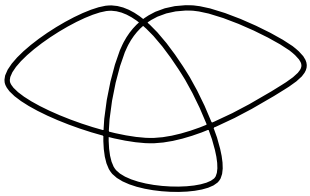


Fig. A trivial knot.

many ideas of string theory have been adopted by mathematicians. The theory has nourished some mathematical disciplines like algebraic geometry, so string theory is sort mathematical jottings without physical evidence.

Loop quantum gravity

Another idea next to string theory and which is considered as a rival to it, with the same main target to find a quantum theory of gravity, and which is also very controversial is: loop quantum gravity (QLT). The theory came into existence in the year 1986 by Abhay **Ashtekar**$(1949-)$ an Indian physicist working in the USA. Thus the loop quantum gravity is seen as a new theory that tries to unify quantum mechanics and general relativity. The idea is to write general relativity in the language of gauge theory, but this time to consider the space time as composed of *loops* . The collection of these loops constitute what is called a *spin networks*.

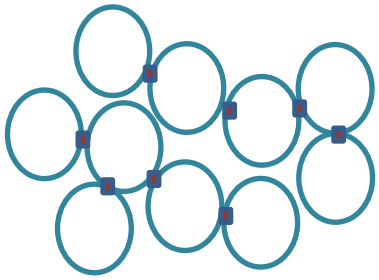


Fig. A network of loops

So in order to quantify spacetime in general relativity, we quantify geometry, we then consider it as a discrete space of loops. This has led to the idea of loops and the system of loops linked to each other. It has been suggested by the proponents of this theory, that gravity is a network of loops interconnected to each other by nods. Some developments were added in the $1990s$ by American Lee **Smolin**$(1955-)$ and Italian Carlo **Rovelli**$(1956-)$ that suggested the implication of the **Wheeler-DeWitt** equation

$$\hat{H}(x)|\psi\rangle = 0$$

And the theory has been ended in 9 dimensional space. Until now, all these theories are only mere speculations.

Exercises

1) Given the Euler integral $B(x,y)$, find $B(y,x)$.

1) is $B(x,y)$ linear ?

1) show that $B(x,y) = B(y,x)$ (symmetry).

1) evaluate

i) $B(0,0)$

ii) $B(1,1)$

iii) $B(1,0)$

iii) $B(0,1)$

1) knowing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, show that $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$.

1) show that $\Gamma(n+1) = n!$.

1) Evaluate $\int_0^{+\infty} e^{-x^2} dx$ (use $\Gamma(x)$).

1) knowing $\int_0^{+\infty} e^{-x^2} dx$, evaluate $\int_{\mathbb{R}} e^{-x^2} dx$.

OPEN PROBLEMS

We must not believe those, who today, with philosophical bearing and deliberating tone, prophesy the fall of culture and accept the ignorabimus. For us there is no ignorabimus, and in my opinion none whatever in natural science. In opposition to the foolish ignorabimus, our slogan shall be : we must know, we will know!

David **Hilbert**

David **Hilbert** was a prominent German mathematician of his age and a very active researcher, he was considered as a universal mind like his contemporary French Henri **Poincaré**, making research in almost all the branches of mathematics of his time. Although not a physicist by formation, we know that he exceled in theoretical physics as well, he for instance arrived at the equations of gravity as did **Einstein**, he participated in knowledgeable discussions through his lectures as in the case of general relativity and quantum theory[65]. He even gave lectures on quantum theory during the winter semester of $1926-1927$ at the University of Gottingen.

Hilbert is also known for his trend to logic and to axiomatization of mathematics, as with his book *Foundation of Geometry*(1899). Also he is considered the father of mathematical formalism. His investigations freed the road to found modern mathematical logic and to the important work in **Gödel** and **Turing**.
Through his interest in all aspects of the mathematical science, he managed to gather a list of 23 mathematical open questions, the most important to his eyes at the beginning of the XX century. At the second International Congress of Mathematics held in Paris in august 1900, **Hilbert** proposed the Problems to his mathematical community. Those problems are known in the history of mathematics as *Hilbert Problems*.

[65] See : *David Hilbert's Lectures on the Foundations of Physic* $1915-1927$(Sprinnger-2009).

The sixth problem

The Sixth of Hilbert Problems was of a physical nature, it asked for an axiomatization of physics, which mean the axiomatization of all existent physical theories. Hilbert words go like this:

…Important investigations by physicists on the foundations of mechanics are at hand; I refer to the writings of Mach, Hertz, Boltzmann and Volkman. It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also….conversely one might try to derive the laws of motion of rigid bodies by a limiting process from a system of axioms depending upon the idea of continuously varying conditions on a materiel filling all space continuously, these conditions being defined by parameters. For the question as to the equivalence of different systems of axioms is always of great theoretical interest.

In mathematics to axiomize(or axiomatize) a theory means to found it on solid true statements known as *axioms*. The idea began with Greek **Euclid**, he was the firsts to give a list of geometrical axioms that he saw were "evident" , then worked out the system to infer logically with proofs all the theorems of geometry known in his time from the axioms[66]. So the first science to be axiomized was Euclidian geometry. **Hilbert** continued

…If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large as a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories.

Theories, other than geometry, which are founded on axioms : set theory, natural number system, algebraic topology,…We know that quantum mechanics has been formalized using **Hilbert** spaces by one of the student of **Hilbert** himself. He put all the system on three main axes:

- physical axioms
- analytic expressions.
- physical meaning or interpretation.

Another work using axiomatic foundations was the big work of **Kolmogorov**$(1903-1987)$ when he formulated the theory of probability by using measure theory. But is it really

[66] **Hilbert** continued the work of Euclid by giving the necessary and sufficient axioms for Euclidian geometry in his known book : The *Foundations of Geometry*(1899).

possible to axiomatize physics like mathematics? Mathematics is very different from all sciences. In fact it is the language of physics, as well as other different branches of science. Mathematics is an ***abstract*** and exact science, using formal proofs, what is true at the time of Euclid is true now. Whereas physics is a materialistic nature that its results change from time to time, the phenomena of physics are not predictable.

The sixth problem was really quite strange at the time, it was proposed before the development of quantum physics of the 20th century. As specialists say[67], the project was not very successful. We can see three different opinions on the sixth Hilbert problem :

- it is vague.
- it is obsolete.
- it is open.

Perhaps the main common feature between mathematics and physics is the use of probability in physical theories. Before 1900, there is for example probability occurring in the statistical mechanics, and afterwards quantum mechanics used probability extensively by considering the module of the wave function.

The nature of physics problems

Finding solutions for problems is the main task of scientists in all the scientific disciplines, it is in this way that there is development in all aspects of life. But usually many of the problems can be subject to speculations, sometimes metaphysical, so we have two types of problems:

- Physics problems: that can be solved using mathematics tools.
- Philosophical questions: Those are not technical, so they cannot be solved with the tools off mathematics. We can also say that they are more philosophical than physical. Sometimes they are given in the form of principle without evidence like the principle.

So open problems in physics abound, but here I myself to the problems of theoretical physics that are in a way have link to relativity, quantum theory or cosmology. One of the very famous problems in XX century is the fine tuning.

[67] See : Newton **Costa** and Francisco **Doria**: *David Hilbert's Sixth Problem* (Springer-2022).

The fine tuning of the universe

In 1916, German Arnold **Sommerfeld** introduced a constant relating e, h and c to quantify the electromagnetic field , its expression is

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}$$

This called the *structure constant*. It has no unity(dimensionless) and it measures the strength of electromagnetic force between charges like electrons.

In fact there are a lot of physical constants that led physicists to wonder why they have exactly the values they have. It has found out that if those constants were less or more than their real values, the universe itself would not be the same. This led cosmologists to assert that the universe is tuned for our existence, so that is the *fine tuning* mystery

Many physical constants have the exact values they have to permit life and let us to exist in the universe, so the universe is fine tuned for life.

Below are six examples of these constants :

Constant	Value
ε nuclear efficiency	0.007
Λ cosmological constant	0.7 ?
Ω Cosmic number	1
D dimension	3
Q ratio of energies	0.00001
N strength of nuclear force	10^{12}

So there is a fine tuning of the universe, otherwise we are not here[68]. The history of the universe and how it originated and the past of its development is called *cosmology*. We have seen some of this with the dynamics of universe and its expansion in the theory general relativity. It appears that these constants have been tuned for the Big Bang and for

[68] See for example the book of English Martin **Rees** *: Just Six Numbers* (1999).

the beginning of the universe. So fine-tuning enabled the existence of the universe and our existence too, this tuning has enabled life to develop on earth.

Usually scientists ask in this domain many questions:

- What do we know about the universe, and how do we know it?
- Where did the universe come from, and where is it going?
- Did the universe have a beginning?
- if so, what happened before then?
- How and why did it begin?
- Will it come to an end, and if so, how?[69]

Problems

Physical problems

- What is an electric field(a magnetic field) ?

- Is there a magnetic monopole ?

- What is the mechanism of high-temperature superconductivity ?

- The mathematical model for the Yang-Milles theory.

- Do neutrinos decay ?

- What are dark matter and dark energy?

[69] Stephen **Hawking**: *A Brief History of Time*(1988).

—Why is the strong force a short-range force?

—Do neutrons have antiparticles ?

—Does Higgs boson have an antiparticle ?

—The problem of confinement.

—Do neutrons have magnetic moment ?

—The problem of mass.

—The hierarchy problem.

—Why is the quantity of antimatter lesser than the quantity of matter?

—is there a law that unify all four forces of nature(theory of everything)?

—is there a quantum theory for gravity?

—What is the number of flavors do neutrons have ?

—Why the Standard Model has the patterns it has?

—Does the Hawking radiation(black hole radiation) exist ?

—Does the graviton exist ?

—Does a black hole produce thermal energy ?

—What is the mass of the neutrino ?

—The **Penrose** conjecture(cosmic censorship hypothesis).

—What is the origin of the mass for the neutrino ?

—What is life-time of the neutrino ?

—**Hawking** conjecture(chronology protection conjecture).

— The BKL conjecture(general relativity).

— What is the value of the Hubble constant?

— What is the exact age of the Universe?

Phylosophical problem

—Is time travel possible?

—Are there higher dimensions?

—Why did the Big Bang take place at the time it did?

—Where did the universe come from, and where is it going?

—Did the universe have a beginning?

—if so, what happened before then?

—How and why did it begin?

—Will it come to an end, and if so, how?

—Are there parallel universes?

—What is empty space(perfect vacuum) made of ?

—is the universe finite or infinite ? how can we prove that ?

—why black holes are formed away from our planet ?

—Is there an extraterrestrial life (related to Fermi paradox) ?

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